

The Bloch Principle

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We formulate and prove an optimal version for quasi-projective surfaces of A. Bloch's dictum, "Nihil est in infinito quod prius non fuerit in finito" by way of a complement to a theorem of J. Duval.

Introduction

In [Bl], A. Bloch, obtained a substantive generalisation of Montel's theorem (maps of the disc to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are normal) to the case of $\mathbb{P}^2 \setminus \{4 \text{ lines}\}$, with H. Cartan's thesis, [C], being the same in arbitrary dimension, *i.e.* $\mathbb{P}^n \setminus \{n+1 \text{ planes}\}$, $n \geq 3$, or, perhaps, $n \geq 2$ given a missing proof of a lemma in the former which Cartan provided. The precise statement is as follows: choose a basis X_i of a $n+1$ dimensional vector spaces, and thus identify the simple normal crossing divisor, B , consisting of $n+2$ planes in \mathbb{P}^n in general position with the coordinate hyperplanes $X_i = 0$ together with the hyperplane $X_0 + \cdots + X_{n+1} = 0$, similarly for every $I \subset \{0, \dots, n+1\}$, $2 \leq |I| \leq n$ there are diagonal hyperplanes $\Lambda_I = \sum_{i \in I} X_i = 0$, every $\Lambda_I \setminus B$ is covered by \mathbb{G}_m 's, and the union Z

of the Λ_I is, an identification of the smallest Zariski closed subset $Z \subset \mathbb{P}^n$, such that (Theorem of Borel) every non-trivial (a precision which will, henceforth be eschewed) entire map $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus B$ factors through Z , then a sequence of holomorphic discs $f_n : \Delta \rightarrow \mathbb{P}^n \setminus B$ which is not (compact-open sense) arbitrarily close to Z admits a subsequence converging uniformly on compact sets.

Bloch, [Bl], makes his pleasure in the theorem quite clear, and, arguably with good reason, since it is way more difficult than the relatively trivial assertion and/or exercise in the definition of the Wronskian that every entire map $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus B$ factors through Z . The latter is, of course, for $n = 2$, the motivating "infinito" to which his dictum referred, while the uniform convergence for discs not arbitrarily close to Z is the "in finito". Nevertheless, for a very long time this theorem of Bloch/Cartan's thesis was wholly sui generis even to the point of there being no other non-obvious (*i.e.* not negatively curved or similar) examples of quasi-projective varieties (X, B) (so, inter alia, projective varieties X if the boundary B is empty), such that one could assert that the property, **Infinito** *There is a proper Zariski closed subset $Z \subset X$ (implicitly without generic points in B) such that any entire map $f : \mathbb{C} \rightarrow X \setminus B$ factors through Z .* was occasioned by a stronger theorem "in finito", such as normal convergence for sequences of discs not arbitrarily close to Z . To some extent this changed with the appearance of Brody's lemma, [B], *i.e.* for X not just projective, but compact, infinito holds with $B = Z = \emptyset$ iff every sequence of holomorphic discs

$f_n : \Delta \rightarrow X$ has a uniformly convergent sub-sequence, together with Mark Green's complement, [G], in the presence of a boundary, *i.e.* everything as above, so again $Z = \emptyset$, and furthermore if $B = \sum_i B_i$ is a sum of Cartier divisors then for every set of components I , no stratum $B_I := \cap_{i \in I} B_i \setminus \cup_{j \notin I} B_j$ admits an entire curve. Consequently although Brody's lemma, and modulo the boundary restrictions, Green's complement, turned Bloch's principle into a theorem for $Z = \emptyset$ the Bloch-Cartan theorem remained *sui generis* as the only example of the Bloch principle with a non-empty exceptional set Z .

The first step to new examples occurred when infinito was established, [M1], for surfaces, X , of general type with $s_2 = c_1^2 - c_2 > 0$. However, under such hypothesis, unlike the Bloch theorem, $Z \setminus B$, may admit, projective, \mathbb{P}^1 , or, affine lines, \mathbb{A}^1 , and should this occur there are sequences of discs $f_n : \Delta \rightarrow X \setminus B$ which are not arbitrarily close to Z but which do not converge uniformly on compact sets. Such discs are, however, rather small, and, appropriately understood do actually converge, *i.e.* not uniformly, but in the sense of Gromov, to a disc with bubbles. Indeed, many so called "counterexamples" to the Bloch principle are of this type, which leads us to pose the best possible variant,

In finito *A sequence of holomorphic discs $f_n : \Delta \rightarrow X \setminus B$ not arbitrarily close to $Z \cup B$ (or possibly just Z) has a subsequence converging to a disc with bubbles.* so that, conjecturally, [M3], the Bloch principle becomes infinito iff in finito, and, this was established for the said class of surfaces in [M2], which, in turn, apart from the Bloch-Cartan theorem, constituted not only the unique examples of the Bloch principle with $Z \neq \emptyset$, but had a feature that Bloch-Cartan does not, *viz.* the necessary appearance of bubbles in finito.

Of course, [M2], gave a proof of the Bloch theorem for $\mathbb{P}^2 \setminus \{4 \text{ lines}\}$ en passant, and, although cleaner, neither here, nor in the study of quasi-projective surfaces $X \setminus B$ with $s_2 > 0$ did it liberate itself from the same methodological flaw of Bloch-Cartan, *i.e.* rather than attempt to prove infinito implies in finito directly, one was simply doing the infinito proof for the corresponding class of varieties better, and, as it happens, with a similar, albeit reversed, inductive structure to Bloch-Cartan, with a view to proving the in finito assertion directly. Worse, albeit this is properly discussed below, such a variant of the Bloch-Cartan strategy can only work in the presence of optimal theorems on the variation in the normal direction of discs which are approximately solutions of certain O.D.E.'s. This holds for $s_2 > 0$, [Br], but may fail otherwise, [M4].

As such, the methodology was clearly un-sustainable, and the key development permitting one to go directly ex infinito in finitum is a much better version of Brody's lemma due to Julien Duval, [D], to wit, for X compact, let $f_n : \Delta \rightarrow X$ be a sequence of discs defined up to the boundary such that the ratio of the length of the same to the area goes to zero, then, for A the (more correctly a up to subsequencing) resulting closed positive (Ahlfors) current: if A has mass on a compact set $K \subset X$, then there is an entire map with bubbles cutting K . This is a much more finely tuned instrument than [B], *e.g.*, already for $Z = B = \emptyset$, it gives not just infinito iff in finito but, [D], iff Gromov's isoperimetric inequality. Another corollary, was that [M2] could supersede itself, *i.e.* the ideas therein, and, in fact, the O.D.E. free ones, could be used to establish

the Bloch principle for any quasi-projective surface. This, modulo bearing in mind that op. cit. is a pre-print, so, we repeat some of its contents to ensure no logical appeal to it, is the content of the present, which we may now summarise.

In the first place, contrary to the assertion in the introduction of [D], it is not true that McQuillan theory uses closed currents in the sense of Ahlfors, 1.5, but rather in the sense of Nevanlinna, 1.6. Alternatively it chooses a point, §1.1, on a Riemann-surface should this have a border. This choice is, ultimately just a means to an end, *i.e.* establishing choice free theorems, but it is essential in bringing algebraic geometry into play by way of 1.1.2, so that in contradistinction to “usual Nevanlinna theory” the term at the origin goes from the least to the most important. This also poses problems, the first of which is to extract closed positive currents in the Nevanlinna sense from discs, which is sufficiently more difficult than the Ahlfors sense, that we’ve only done it, 1.2.4, under the hypothesis of projectivity of the target. Similarly we need Duval’s theorem in the Nevanlinna rather than the Ahlfors sense, albeit we reduce the former to the latter in 1.2.10 and 1.3.10. Such considerations are enough to get us to the starting line if $B = \emptyset$, *i.e.* if infinito holds but in finito fails there is a closed, and without loss of generality nef., positive current T arising from discs, in the Nevanlinna sense, supported on Z . In the case of a non-empty boundary, however, one must give conditions that bubbles do not form in the boundary, 1.3.7, which in the first approximation amounts to Green’s condition on strata above but for \mathbb{A}^1 ’s rather than \mathbb{C} ’s, 1.3.8. Such a condition holds in an almost étale neighbourhood of the boundary of the minimal model of any quasi-projective surface which is not covered by \mathbb{A}^1 ’s, but such an almost (étale) boundary condition does not imply no-bubbling in the boundary (there are counterexamples) whereas the condition of being a minimal model does, 2.1.3, which, again, since it’s a bit subtle, we only prove in the quasi-projective case, even though it has a perfect almost complex sense. This suggests, and, it’s necessary, that Bloch’s principle be formulated on the minimal, or better canonical model.

At this juncture there are some subtleties, which, in turn, are complicated by the currently overwhelming tendency in the study of complex hyperbolicity to treat the canonical model, if at all, in qua, rather than ex qua. For example, the Green-Griffiths conjecture, *i.e.* infinito holds iff general type, is posed for entire maps to surfaces of general type, rather than the more conformally natural hypothesis of entire maps to the Vistoli covering champ (orbifold), 2.1.1, of their canonical model, which, after all, is where its Kähler-Einstein metric is defined. As such, although the Bloch principle has been formulated in 2.1.2, and proved in 2.2.4, in a way that wholly responds to this in qua tendency, it is most easily summarised ex qua. More precisely, if a pair (S, B) with log canonical singularities, and $K_S + B$ ample is given, then the singularities of S are either quotient or elliptic Gorenstein. The latter, P , say, are what one finds at cusps of ball or bi-disc quotients, and in a neighbourhood of $V \supset P$, the Kähler-Einstein and Kobayashi metrics on $V \setminus P$ are complete and mutually comparable. As such, notation notwithstanding, these should always be treated as belonging to the boundary, which might be better written $B \cup P$. Amongst the quotient singularities, Q , however, those in the interior $S \setminus B$ do not naturally (ex qua), cf.

2.1.2, belong to the boundary, so we replace S by a champ de Deligne-Mumford, $S \rightarrow S$ smooth over Q , and, otherwise isomorphic to S on which we have,

Main Theorem (2.2.4) *Notations as above, then the following are equivalent,*
 (Infinito) *There is a proper Zariski closed $Z \subset S$ such that every entire map $f : \mathbb{C} \rightarrow S \setminus \{B \cup P\}$ factors through Z .*

(In finito) *A sequence of holomorphic discs $f_n : \Delta \rightarrow S \setminus \{B \cup P\}$ (equivalently, [CHO], $f_n : \Delta \rightarrow S \setminus \{B \cup P \cup Q\}$) which is not arbitrarily close (compact open sense) to Z has a sub-sequence converging to a disc with bubbles in S and, 1.3.7, without bubbles in the boundary.*

Which, if one likes things in qua, *i.e.* one starts, 2.1.2, from a smooth surface \tilde{S} with simple normal crossing boundary \tilde{B} of which (S, B) is the canonical model, then the appropriate statement, 2.2.4, is obtained on replacing the champ S by another, S' , according to the rule: S' is isomorphic to S around $q \in Q \cap (S \setminus B)$ if the pre-image of q is a connected component of \tilde{B} , and isomorphic to S around q otherwise. Irrespectively, one should be aware that the above is not simply theoretical, *i.e.* while as noted, [M2], provides another (more difficult) proof of in finito whenever $s_2(X, B) > 0$, there is no similar proof otherwise, but there is a proof, [M5], of infinito under hypothesis such as $13c_1^2 > 9c_2$.

To some extent, the above theorem constitutes the limit of our present ambition, but one should always have an eye towards trying to understand the geometry of algebraic surfaces as well as we understand Riemann surfaces or complete 3-manifolds. Ultimately, one might hope, [M3], that this involves some sort of uniformisation theorem defined via extremal discs. In the first instance, however, it involves Kobayashi's intrinsic metric, which by the above, in the presence of infinito, is a complete metric off Z , so that, around this exceptional set of rational and elliptic objects, we should aim to quantify its degeneration. Such quantification might, therefore, be considered a crypto Bloch principle. Inevitably, it must involve a resolution of singularities of $(S, Z + B)$, so we defer to 2.3.6 for the exact statement. The key feature, however, is what occurs at a generic point of Z , where, by way of notation, we take x to be a normal coordinate to Z , y longitudinal, and find for every $\delta > 0$ a constant $c(\delta) > 0$, such that a lower bound for the Kobayashi metric close to Z is given by,

$$\frac{c(\delta)}{\log^4 |x| (\log |\log |x||)^{1+\delta}} \cdot (dx \otimes d\bar{x} + |x|^2 dy \otimes d\bar{y})$$

which is very close to best possible, but at least in the normal direction appears to be slightly out by the scaling factor $(\log |x|)^{-4} (\log |\log |x||)^{-1-\delta}$.

Finally, let us say something about the proof. To the preliminaries already encountered, is to be added, in all dimensions, some tautologies, 1.4, about what happens on taking (logarithmic) derivatives of discs in say, $\mathbb{P}(\Omega_X^1(\log D))$, for D a simple normal crossing divisor. Roughly speaking, this amounts to finding a derived current T' pushing forward to a current T defined by discs on X , whose intersection with the tautological bundle L on $\mathbb{P}(\Omega_X^1(\log D))$ satisfies $L \cdot T' \leq D \cdot T$, albeit there are a series of much more precise statements 1.4.1-1.4.3, involving pointed Riemann surfaces, a complete metric on $X \setminus D$, the multiplicity free

intersection with D , and the distance of the point from D . Of all the preliminary sections §1.1- §2.1, the surprise for the necessity of which at the beginning of the 21st century notwithstanding, probably only this tautological one, §1.4, can be considered remotely definitive- other key preliminary propositions 1.2.4, and 2.1.3 having recourse to the crutch of projectivity. This said, let us come to the point, once the (complex) dimension exceeds 1 there is, contrary to some false and bogus conjectures in circulation, absolutely no way to get from the tautological upper bound on the tautological degree implied by our closed formula, 1.4.2, and the length area principle to an identical bound for the (log) canonical degree. The sense here, however, of “dimension 1” should be very broadly interpreted. For example in the case of the Bloch-Cartan theorem it means discs limiting on the space of lines in \mathbb{P}^n viewed as a n th order ODE, or, equivalently sub-variety of the n th jet space, or solutions of a first order O.D.E. in the case of surfaces with $s_2 > 0$. In situations such as these where the discs limit on a sufficiently nice O.D.E. one can bound the canonical by the tautological degree, or, more or less equivalently, the sectional curvature (understood at the weaker Nevanlinna level) is at most the Ricci curvature along the solutions of the O.D.E., albeit the arguments, be it [Bl], [C], or [M2] can get rather involved. Thanks to Duval’s theorem, however, and 2.1.3, the only O.D.E.’s obstructing ex infinito in finitum have order zero, and are better referred to as algebraic curves, and we have,

Main Lemma (2.2.1) *Let T be a closed positive current on an algebraic surface, X , afforded by (pointed) discs in the Nevanlinna sense, 1.6, such that,*

(a) *T is supported in a simple normal crossing divisor D .*

(b) *The origins of the discs do not accumulate on D .*

then for T' on $\mathbb{P}(\Omega_X^1(\log D))$ the current associated to the logarithmic derivatives of the discs, and L the tautological bundle on the same,

$$L \cdot T' \geq (K_X + D) \cdot T$$

With the content of the proof being the interplay between the hypothesis (b) and adjunction. The above main theorem/ Bloch principle, and its in qua variants, appropriately, and necessarily (to repeat: in all cases the principle requires a minimum of minimality of the model) corrected for quotient and elliptic Gorenstein singularities is, 2.2.4, an immediate corollary of the lemma. The lemma has, however, further utility, and is, for example, the reduction that the programme, [M5], for resolving the Green-Griffiths conjecture aims for.

Thus, while the regrettably long preliminaries of §1 are valid in all dimensions, their application to the Bloch principle in §2 is a surface theorem, which, inter alia, and much after the event, permits, via [D], Bloch’s theorem to be considered trivial. One is no closer, however, to a similar statement in all dimensions. In the quasi-projective setting there are, in all probability, examples where no model affords a no bubbling in the boundary lemma à la 2.1.3, while even in the projective case, large discs defining a current supported in a proper sub-variety isn’t much information when this has dimension more than 1. As such, Cartan’s thesis has a particular inductive (in the dimension) structure which is not true in higher dimension, and which causes it to remain sui generis.

1 Nevanlinna Theory

1.1 Area and double integration

Throughout this section (Σ, p) will be a pointed Riemann surface with boundary. The boundary will always be understood to be regular, *i.e.* it admits a Green's function, or better, for the present, a psh. function $g : \Sigma \rightarrow [-\infty, 0]$ vanishing on the boundary, and finite on $\Sigma \setminus \{p\}$ satisfying:

$$dd^c g = \delta_p \quad (1.1)$$

and we denote by z a function around p such that,

$$g = \log |z|^2 + O(|z|) \quad (1.2)$$

For $r \in (-\infty, 0]$, let Σ_r be the open set of points where $g < r$. This is relatively compact for $r < 0$, and we may integrate, say:

$$\int_{\Sigma_r} : A^{1,1}(\Sigma) \rightarrow \mathbb{C} : \omega \mapsto \int_{\Sigma_r} \omega \quad (1.3)$$

The choice of the point p permits Nevanlinna's variant on this,

$$\oint_{\Sigma_r} : A^{1,1}(\Sigma) \rightarrow \mathbb{C} : \omega \mapsto \frac{1}{2} \cdot \int_{-\infty}^r dt \int_{\Sigma_t} \omega \quad (1.4)$$

And of course we extend these definitions whether to $r = 0$, or ω just a signed measure, or both, provided they continue to have sense as absolute integrals.

The exact relation of Nevanlinna's definition with geometry is unclear- after all it involves a choice. It is, therefore, perhaps best to treat it as a technical tool whose utility comes from:

Fact 1.1.1. Let \bar{D} be a metricised Cartier divisor on Σ , with $\mathbb{I}_D \in \Gamma(\mathcal{O}_\Sigma(D))$ the tautological section, then,

$$\begin{aligned} \oint_{\Sigma_r} c_1(\bar{D}) &:= \sum_{-\infty < g(z) < r} \text{ord}_z(D) \cdot \frac{(r - g(z))}{2} + \text{ord}_0(D) \frac{r}{2} \\ &\quad - \int_{\partial \Sigma_r} \log \|\mathbb{I}_D\| \cdot d^c g + \lim_{z \rightarrow p} \log \frac{\|\mathbb{I}_D\|}{|z|^{\text{ord}_0(D)}} \end{aligned}$$

Proof. We're always working in a complex setting, so by definition: $c_1(\bar{D}) = -dd^c \log \|\mathbb{I}_D\|^2$, almost everywhere, after which we have an exercise in integrating by parts. \square

This leads to the principle application of 1.4, which doesn't have a parallel under 1.3, to wit:

Example 1.1.2. Let X be a compact complex space, or indeed an analytic champ de Deligne-Mumford, and \bar{D} a metricised divisor on it, then for any map $f : \Sigma \rightarrow X$ from any (pointed) Riemann-surface such that $f(p) \notin S$, and any r ,

$$\int_{\Sigma_r} f^* c_1(\bar{D}) \geq \log \|\mathbb{I}_D\|(p) - \sup_X \log \|\mathbb{I}_D\|$$

The supremum in question depends only on X and D , we may as well normalise so that it's zero, so the only obstruction to a positive intersection in the Nevanlinna sense of f with D is the distance of $f(p)$ to D .

Here is another example of the utility of 1.1.1,

Example/Definition 1.1.3. Suppose that Σ is hyperbolic, and let ω be the Poincaré metric. The form ∂g is meromorphic, and the (signed) sum of its zeroes and poles in any Σ_r is the Euler characteristic $\chi(r)$ of the same- just consider the compact double obtained by Schwarz reflection in the boundary. The Nevanlinna variant of the Euler characteristic is given by,

$$E_\Sigma(r) := \frac{1}{2} \cdot \int_{-\infty}^0 \chi(\Sigma_t \setminus p) dt + \frac{r}{2}$$

and the definition of constant curvature -1 implies, for ∂ a field dual to ∂g , and z as in 1.2,

$$\int_{\Sigma_r} \omega + E_\Sigma(r) + \log \left\| \frac{\partial}{\partial z} \right\|_\omega(p) = \int_{\partial \Sigma_r} \log \|\partial\|_\omega d^c g \leq \log[(\text{length}_\omega)(\partial \Sigma_r)]$$

which, although there are more interesting length-area isoperimetric inequalities using 1.3 rather than 1.4, their proofs are much more difficult.

1.2 Closedness and positivity

Let us consider in more detail the situation presented in 1.1.2, with ω a positive $(1,1)$ form on X , and $f_n : \Sigma_n \rightarrow X$ maps from (pointed) Riemann surfaces. The alternatives 1.3 & 1.4 as to how we integrate over Σ_n lead to competing definitions for bounded currents on X , viz:

$$A_n(r) : A^{1,1}(X) \rightarrow \mathbb{C} : \tau \mapsto \left(\int_{\Sigma_{n,r}} f_n^* \omega \right)^{-1} \int_{\Sigma_{n,r}} f_n^* \tau \quad (1.5)$$

and its Nevanlinna counterpart:

$$T_n(r) : A^{1,1}(X) \rightarrow \mathbb{C} : \tau \mapsto \left(\int_{\Sigma_{n,r}} f_n^* \omega \right)^{-1} \int_{\Sigma_{n,r}} f_n^* \tau \quad (1.6)$$

Where in either case we will implicitly suppose that there is $r \in (-\infty, 0)$ such that as $n \rightarrow \infty$:

$$\int_{\Sigma_{n,r}} f_n^* \omega \rightarrow \infty, \text{ respectively, } \int_{\Sigma_{n,r}} f_n^* \omega \rightarrow \infty \quad (1.7)$$

and one should bear in mind that we are not asserting any a priori relation between the respective conditions. Under the former of these conditions, and up to moving r a little, the length area principle implies, without much work, that a subsequence of the $A_n(r)$ converges to a closed positive current. The corresponding statement for $T_n(r)$ is more difficult, and we'll limit our discussion to the case where all the f_n are maps from the unit disc, Δ , pointed in the origin. In such a situation it is convenient to change the notation according to,

Warning 1.2.1. Should we be discussing uniquely sequences of discs then $\Delta(r)$ will be the (pointed) disc of radius r . The solution of 1.1 is, of course, $\log |z|^2$, so for τ (1, 1) on the disc:

$$\oint_{\Delta_r} \tau = \int_0^r \frac{dt}{t} \int_{\Delta(t)} \tau \quad (1.8)$$

We will further limit our attention to X projective, so in the first instance \mathbb{P}^n for some n , and we make,

Defintion/Choice 1.2.2. A Fubini-Study metric on \mathbb{P}^n depends on a choice of basis. Make such a choice and take the ω occurring whether in 1.5 or 1.6 to be the resulting Fubini-Study form.

Which we will suppose in,

Lemma 1.2.3. Let $x \in \mathbb{P}^n$, then there is an open (archimedean) neighbourhood $U \ni x$, and constants c_1 , c_2 , and $N > n$, depending only on U and the choice of 1.2.2 such that for every $f : \Delta \rightarrow \mathbb{P}^n$, and $0 < r < 1$.

$$R \|df(z)\|_\omega \leq c_1 \oint_{\Delta(R)} f^* \omega + c_2$$

for $2|z| < R$, where $R = \min\{\frac{r}{N} \exp(-\oint_{\Delta(r)} f^* \omega), r\}$, and the implied norm on the disc is just the Euclidean one.

Proof. Everything is homogeneous under the unitary group so we may as well say $x = [1, \dots, 1]$, in the basis X_i generating $\mathcal{O}_{\mathbb{P}^n}(1)$ and affording ω . A disc is Stein, so fixing a trivialisation of $f^* \mathcal{O}_{\mathbb{P}^n}(1)$ allows us to write $f = [f_0, \dots, f_n]$ in these coordinates for f_i functions on the disc without common factor. As such if H_i is the i^{th} -coordinate hyperplane, we may suppose $U \cap H_i = \emptyset$, $\forall i$, so:

$$\oint_{\Delta(r_0)} f^* \omega = - \int_{|z|=r} \log \frac{\|f^* X_i\|}{\|f^* X_i\|(0)} \frac{d\theta}{2\pi} + \sum_{0 < |z| < r} \text{ord}_z(f^* H_i) \log \frac{r}{|z|}$$

Now for convenience put, $T_f(r)$ to be the left hand side of the above, and take U sufficiently small so that the infimum over $u \in U$ and i of $\|X_i\|(u)$ is $N^{-1} > 0$. Whence if $f(z) \in H_i$ with $|z| < r$, then:

$$|z| \geq \frac{r}{N} \exp(-T_f(r)).$$

So put $R = \min \left\{ r, \frac{r}{N} \exp(-T_f(r)) \right\}$ then for $|z| < R$, $f_i(z) \neq 0$. In particular if we identify affine n -space \mathbb{A}^n with $\mathbb{P}^n \setminus H_0$ then $\Delta(R)$ maps to \mathbb{A}^n under f with standard coordinates $g_i = f_i/f_0$ and each g_i is a unit on $\Delta(R)$.

Consequently Jensen's formula gives,

$$\frac{g'_i(z)}{g_i(z)} = \int_0^{2\pi} \log |g_i(Re^{i\theta})| \frac{2Re^{i\theta}}{(Re^{i\theta} - z)^2} \frac{d\theta}{2\pi}$$

We require to control the modulus of the logarithm in the integrand, so to this end observe that inversion on $\mathbb{G}_m^n = \{x_1 \dots x_n \neq 0\} \subset \mathbb{A}^n$ is well defined on a modification $\rho : W \rightarrow \mathbb{P}^n$ which is an isomorphism outside crossings of hyperplanes. In particular if $i : W \rightarrow \mathbb{P}^n$ is the extension of inversion then,

$$i^* \mathcal{O}_{\mathbb{P}^n}(1) = \rho^* \mathcal{O}_{\mathbb{P}^n}(n) - E$$

for E an effective divisor on W contracted by ρ . Applying this to $if|_{\Delta(R)}$ gives in the notation of 1.1.2:

$$\frac{1}{2} \int_{|z|=R} \log(1 + \sum_{i=1}^n |g_i|^{-2}) \frac{d\theta}{2\pi} \leq nT_f(R) - \log f^* \|\mathbb{I}_E\|(0) + \frac{1}{2} \log(1 + \sum_{i=1}^n |g_i|^{-2})(0)$$

The definition of N implies that the final term at the origin is bounded by $\log(1 + nN^2)^{1/2}$, which in any case is just some constant $C(U)$ depending on U , and such a constant also bounds the exceptional divisor as soon as $\rho(E)$ is not in the closure of U . A similar bound for $\log(1 + \sum |g_i|^2)$ is even easier, and so we obtain, for $|z| < R/2$,

$$\left| \frac{g'_i}{g_i} \right| \leq 8R^{-1} ((n+1)T_f(R) + C(U))$$

with $C(U)$ as per its definition. Now consider the Fubini-Study metric for the particular choice of coordinates, with ω the corresponding $(1,1)$ form then,

$$\omega \leq 2 \sum_i \frac{dd^c |x_i|^2}{(1 + |x_i|^2)}$$

so that plugging in our estimates for g'_i gives the lemma. \square

Let us apply this to establish the non-trivial part of,

Proposition 1.2.4. *Let $f_n : \Delta \rightarrow X$ be a sequence of maps from the disc, and suppose the former (area) alternative in 1.7, holds at some $r_0 \in (0,1)$ with sufficiently rapid growth, cf. 1.2.5, in n (so, always possible after passing to a sub-sequence) then for $r \geq r_0$ outside a set of finite hyperbolic measure (i.e. $(1-r^2)^{-1} dr$) in $(0,1)$ any weak accumulation point of the $A_n(r)$, 1.5, defines a closed positive current. Less obviously, suppose X is projective, the latter (Nevanlinna) alternative of 1.7 holds, and that the $f_n(0)$ converge, then the same conclusion holds for the $T_n(r)$ of 1.6.*

Proof. As we've said, the assertion for the $A_n(r)$ is the length-area principle, and is trivial, *i.e.* for α a smooth 1 form on X ,

$$|\int_{\Delta(r)} f_n^* d\alpha| = |\int_{\partial\Delta(r)} f_n^* \alpha| \leq r \|\alpha\| \int_{\partial\Delta(r)} |df_n|_\omega \frac{d\theta}{2\pi} \leq \|\alpha\| \left(r \frac{d}{dr} \int_{\Delta(r)} f_n^* \omega \right)^{1/2}$$

for a suitable determination of the sup-norm, $\alpha \mapsto \|\alpha\|$ independent of f_n, ω as per 1.2.2, and the rest follows from:

Claim 1.2.5. Let $\delta > 0$ and S_n a sequence of increasing differentiable functions on $[0, 1]$ with $S_n(r_0) \geq n^{2/\delta}$ for some $r_0 \in (0, 1)$ then the set,

$$\{1 > r \geq r_0 : S'_n(r) \geq S_n^{1+\delta}(r) (1-r^2)^{-1}, \text{ for some } n\}$$

has finite measure in the induced hyperbolic measure on the unit interval.

Proof. The set in question has measure bounded by,

$$\sum_n \int_{r_0}^1 \frac{S'_n(r)}{S_n^{1+\delta}(r)} dr \leq \sum_n \int_{S_n(r_0)}^\infty \frac{dx}{x^{1+\delta}} = \frac{1}{\delta} \sum_n \frac{1}{S_n(r_0)} \delta = \frac{\pi^2}{6\delta} < \infty \quad \square$$

As such let's concentrate on getting a similar bound in the Nevanlinna case, which will permit us to apply 1.2.5. Again we begin with Stokes,

$$|\oint_{\Delta(r)} f_n^* d\alpha| = \left| \int_{\Delta(r)} f_n^* \alpha d \log |z| \right| \leq \|\alpha\| \int_{\Delta(r)} |df_n|_\omega \frac{dtd\theta}{2\pi}$$

but now there is a problem close to the origin which requires care, and we put:

$$J_n(s) = \int_{|z|=s} |df_n|_\omega^2 \frac{d\theta}{2\pi}, \quad \text{so that,} \quad \oint_{\Delta(r)} f_n^* \omega = \int_0^r J_n(s) s \log \frac{r}{s} ds$$

which we re-arrange as:

$$I_n(r) := \int_0^r J_n(t) t |\log t| dt = \oint_{\Delta(r)} f_n^* \omega + |\log r| r \frac{d}{dr} \oint_{\Delta(r)} f_n^* \omega$$

Consider for a suitable $\varepsilon > 0$, depending on n , to be chosen,

$$\begin{aligned} \int_{r \geq |z| \geq \varepsilon} |df_n|_\omega \frac{dtd\theta}{2\pi} &\leq \int_\varepsilon^r J_n(t)^{1/2} dt \\ &\leq \left(\int_\varepsilon^r t |\log t| J_n(t) dt \right)^{1/2} \left(\int_\varepsilon^r \frac{dt}{t |\log t|} \right)^{1/2} \\ &\leq I_n^{1/2}(r) \log^{1/2} \frac{|\log \varepsilon|}{|\log r|} \end{aligned}$$

At which point take $r \geq r_0$, n sufficiently large, and use the \ll notation for inequality up to a constant that depends on what we can fix, *i.e.* r_0, X, ω etc.

but definitely not f_n or r . Then by 1.2.3, and in the notation of the same, for any $\log \varepsilon \ll -T_{f_n}(r_0)$ there is a constant, C , independent of r and n such that:

$$\begin{aligned} \int_{|z| \leq r} |df_n|_\omega \frac{dt d\theta}{2\pi} &\ll I_n^{1/2}(r) \log^{1/2} \frac{|\log \varepsilon|}{|\log r|} + \varepsilon T_{f_n}(r_0) \exp(CT_{f_n}(r_0)) \\ &\ll I_n^{1/2}(r) \log^{1/2} \left(\frac{T_{f_n}(r)}{|\log r|} \right) \end{aligned}$$

after a suitably choice of ε depending on n . It only remains to show that $I_n(r)$ is not that much bigger than $T_{f_n}(r)$, which is again 1.2.5. \square

This is the main lemma of this section, on which we'll need some variants, *viz*:

Scholion 1.2.6. Starting from a given projective variety X , we will have need to work with various auxiliary (projective) varieties $\pi_i : X_i \rightarrow X$, $i \in I$ countable, or, indeed champs de Deligne-Mumford with projective moduli, albeit the latter is only a technical convenience that one may eschew. Consider attempting to lift the currents, $T_n(r)$ of 1.6 to currents $T_{in}(r)$ on X_i . The maps π_i will be proper, so we have commutative diagrams:

$$\begin{array}{ccc} C_{in} & \xrightarrow{f_{in}} & X_i \\ p_{in} \downarrow & & \pi_i \downarrow \\ \Delta & \xrightarrow{f_n} & X \end{array}$$

with C_{in} smooth (possibly a champ if X_i is) and p_{in} proper and finite. In order to maintain functoriality, we don't exhaust the C_{in} by their Green's function as in 1.4, but by p_{in} . Thus for $r \in (0, 1)$, $C_{in}(r) = p^{-1}(\Delta(r))$, and,

$$\oint_{C_{in}(r)} : A^{1,1}(C_{in}) \rightarrow \mathbb{C} : \tau \mapsto \frac{1}{\deg(p_{in})} \cdot \int_0^1 \frac{dt}{t} \int_{C_{in}(r)} \tau \quad (1.9)$$

$$T_{in}(r) : A^{1,1}(X_i) \rightarrow \mathbb{C} : \tau \mapsto \left(\oint_{\Delta(r)} f_n^* \omega \right)^{-1} \oint_{C_{in}(r)} f_{in}^* \tau \quad (1.10)$$

Now, by construction, $(\pi_i)_*(T_{in}(r)) = T_n(r)$, but there are the following issues to take care of:

(a) Whether in 1.1.1, 1.1.2, or similar, the point p should be replaced with $p_{in}^{-1}(0)$, so control at the origin for all i and n will still be achievable by control of the whereabouts of the $f_n(0)$.

(b) The $T_{in}(r)$ need not be bounded. Actually this can only happen if π_i is not finite. Indeed, π_i has a Stein factorisation, $X_i \xrightarrow{\tau_i} Y_i \xrightarrow{\sigma_i} X$, or better

$X_i \xrightarrow{\rho_i} |X_i| \xrightarrow{\tau_i} Y_i \xrightarrow{\sigma_i} X$ the (finite) map to moduli followed by Stein factorisation if X_i is a champ, while for any finite map $Y \rightarrow Z$ the difference between a metric on Z and Y is dd^c (bounded), so, for example, when π_i is finite $T_{in}(r)$ is bounded by an exercise in integration by parts à la 1.1.1. If, however, τ_i

were non-trivial, then since everything is projective, a metric ω_i on X_i is almost everywhere of the form:

$$\omega_i = \tau_i^* \eta_i + dd^c \log \|\mathbb{I}_E\|^2$$

for η_i a metric on Y_i and $\|\mathbb{I}_E\|^2$ some distance function on a subscheme E blown down by τ_i . Consequently we can guarantee that the $T_{in}(r)$ are bounded if the origins $f_n(0)$ are bounded away from $\pi_i(E)$, or, just that the distance between them satisfies an appropriate growth condition.

(c) The proof of 1.2.4 could fail, and limits of the T_{in} need not be closed. Modulo a variant, 1.2.7 of 1.2.5, this could only happen if the pre-image of discs under p_{in} of the size for which 1.2.3 holds failed to be discs, which, again, cannot happen provided that the origins $f_n(0)$ are bounded away from the ramification of π_i or satisfy an appropriate growth condition on the distance to it. Better, a further case, therefore, in which one can dispense with any condition on the distance between the ramification in π and $f_n(0)$ is π finite and f_n not meeting any point of the branch locus.

The aforesaid, albeit required, variant of 1.2.5 is simply:

Claim 1.2.7. *Let S_{ni} be a doubly indexed sequence of increasing differentiable functions then for $r \in (0, 1)$ outside a set of finite hyperbolic measure:*

$$S'_{nk}(r) \leq \{S_{nk}(r) + 2e\} \log^{n^2+k^2+1} \{S_{nk}(r) + 2e\} (1 - r^2)^{-1}.$$

Proof. Indeed writing $F_{nk}(r) = S_{nk}(r) + 2e$, we have $\log F_{nk}(0) > 1$, and the set in question has measure bounded by,

$$\sum_{n,k} \int_{F_{nk}(0)}^{\infty} \frac{dx}{x \log^{n^2+k^2+1} x} = \sum_{n,k} \frac{1}{(n^2+k^2)(\log F_{nk}(0))^{n^2+k^2}} < \infty \quad \square$$

Whence to summarise:

Summary 1.2.6.bis: Let $\pi_i : X_i \rightarrow X$ be as in 1.2.6, and T a closed positive current arising from a weak limit of some sub-sequence $T_n(r)$ afforded by discs f_n satisfying the conditions of 1.2.4, then provided the $f_n(0)$ are bounded away from the branch locus of π_i , or, indeed the distance between these satisfies an appropriate growth condition, there is a closed positive current T_i on X_i such that $(\pi_i)_*(T_i) = T$. In addition if D_i is any effective Cartier divisor on X_i such that the $f_n(0)$ do not accumulate in $\pi_i(D_i)$, or, again, a suitable growth condition on the distance between the same, then: $D_i \cdot T_i \geq 0$.

Finally we need to compare the limits afforded by the different definitions 1.5 and 1.6. Observe that in the former case, the limiting discs Δ_n satisfy:

Definition 1.2.8. *The length of the boundary l_n in the metric $f_n^* \omega$ is $o(a_n)$ of the area computed in the same, i.e. if the area alternative of 1.7 holds, the limits of the $A_n(r)$ in 1.2.4 are Ahlfors' currents in the sense of [D, pg. 306].*

Now for $t \in (0, 1)$, let $a_n(t)$ be the area of the disc of radius t with respect to $f_n^* \omega$, and r_0 the supremum of $t \in [0, 1)$ for which $a_n(t)$ is bounded. We'll

suppose that $r_0 < 1$, so the Nevanlinna alternative of 1.7 holds for all $r > r_0$. Fix such a r , then on $[0, r]$ we have probability measures:

$$\begin{aligned} d\mu_n(t) &:= \left(\int_{\Delta(r)} f_n^* \omega \right)^{-1} a_n(t) \frac{dt}{t}, \quad \text{and,} \\ d\nu_n(s) &:= \left(\int_{\Delta(r)} f_n^* \omega \right)^{-1} a'_n(s) \log \left| \frac{r}{s} \right| ds \end{aligned} \quad (1.11)$$

which for any continuous ρ are related by:

$$\int \rho d\mu_n(t) = \int d\nu_n(s) \frac{1}{\log \left| \frac{r}{s} \right|} \cdot \int_s^r \rho(t) \frac{dt}{t} := \int d\nu_n(s) N_r(\rho)(s) \quad (1.12)$$

and the value at s equal to 0, or r , of the Nevanlinna type transform $N_r(\rho)$ is just the above limited in s .

Similarly for $U \subset X$ open, we have the area in U , *i.e.*

$$a_n^U(t) := \int_{f_n^{-1}(U)} f_n^* \omega, \quad \text{together with the ratio: } \phi_n(t) = \frac{a_n^U(t)}{a_n(t)} \quad (1.13)$$

or, if one prefers to keep things smooth, multiply ω by a smooth $[0, 1]$ valued bump function identically 1 on U , and, in any case, there are measures:

$$\begin{aligned} d\mu_n^U(t) &:= \left(\int_{\Delta(r)} f_n^* \omega \right)^{-1} a_n^U(t) \frac{dt}{t}, \quad \text{and,} \\ d\nu_n^U(s) &:= \left(\int_{\Delta(r)} f_n^* \omega \right)^{-1} \left(\int_s^r \phi_n(t) \frac{dt}{t} \right) a'_n(s) ds \end{aligned} \quad (1.14)$$

Subsequencing we may suppose that the ϕ_n converge pointwise to some ϕ , while 1.11 converge to probability measures $d\mu$, $d\nu$ on $[0, r]$, related as in 1.12, and 1.14 converge to measures $d\mu^U$, $d\nu^U$, necessarily absolutely continuous with respect to $d\mu$, respectively $d\nu$. Let us observe:

Lemma 1.2.9. *Notation as above, then:*

- (a) *If $d\mu$ has no support in $(a, b] \subseteq (0, r]$ then $d\nu$ has no support in $(0, b]$. In particular if $r_0 > 0$ then none of the above are supported in $(0, r_0)$.*
- (b) *In the open interval $(0, r) \subset [0, r]$ the Lebesgue derivative of $d\nu^U$ by $d\nu$ is, $N_r(\phi)$, defined exactly as in 1.12, at $s \in (0, r)$.*

Proof. 1.12 is valid for the characteristic function of the interval too, which proves (a), while, by Ergoff's theorem, $N_r(\phi_n) \rightarrow N_r(\phi)$ uniformly on compact subsets of $(0, r)$ which proves (b). \square

From which we progress to:

Fact 1.2.10. Suppose the total mass of $d\mu^U$, equivalently that of $d\nu^U$, is non-zero then if $d\nu^U$ has support on $(0, r)$ there is a $r_0 < t < r$ such that, after subsequencing, $A_n(t)$ of 1.5 converge to an Ahlfors current, non-zero in U . Better still, there is a set, $E \subset (0, 1)$, depending on ϕ_n , of zero Lebesgue measure such that if $r \in (0, 1) \setminus E$ is sufficiently large, then the same holds whenever $d\nu^U$ has support in $(0, r]$.

Proof. The easy case occurs should $d\nu^U$ have support in $(0, r)$ so that by 1.2.9, $\phi|_{(r_0, r)} \neq 0$ on a set of positive Lebesgue measure. Subsequencing doesn't change ϕ , but it does make the measure of the set occurring in 1.2.5, modulo scaling from $(0, r)$ to $(0, 1)$ and putting $S_n = a_n$ as small as we please. Whence we can find $t \in (r_0, r)$ for which we simultaneously have length area for all n on discs of radius t and $\phi_n(t)$ converging to $\phi(t) \neq 0$.

This also proves the better still, with E empty, provided $\phi|_{(r_0, 1)} \neq 0$ almost everywhere, and r is sufficiently large. Consequently we may suppose ϕ zero Lebesgue almost everywhere. For $n \in \mathbb{N}$ consider the decreasing functions:

$$\psi_n(t) := \sup_{m \geq n} \phi_m(t)$$

which still converge to zero Lebesgue almost everywhere. On the other hand, [F, 2.9.19], for Lebesgue almost all r , the limit,

$$\lim_{s \rightarrow r^-} \frac{1}{\log \left| \frac{r}{s} \right|} \cdot \int_s^r \psi_n(t) \frac{dt}{t}$$

exists, and is equal to $\psi_n(r)$, so, zero for $r \notin E$, with E of zero Lebesgue measure. As such, if $\varepsilon > 0$ is given, and $r \notin E$, there is a $\delta > 0$ such that for all $s \in (r - \delta, r)$, $N_r(\psi_n)(s) \leq \varepsilon$. By construction, $N_r(\psi_n) \geq N_r(\phi_m)$ for all $m \geq n$, so for such a r the measure of $d\nu^U$ on $(0, r]$ is at most ε which was arbitrary, *i.e.* the assertion is vacuous at Lebesgue almost all r . \square

The utility of which will present itself in the next section.

1.3 Compactness

Although subordinate to more general theorems, the following will prove useful,

Lemma 1.3.1. Let $f_n : \Delta \rightarrow \mathbb{P}^m$ be a sequence of discs defined up to the boundary with ω as per 1.2.2, and admitting a uniform bound in n for,

$$\int_{\Delta} f_n^* \omega$$

then we may write $f_n = [f_{n0}, \dots, f_{nm}]$ in such a way that after subsequencing the f_{ni} , $0 \leq i \leq m$, converge uniformly to some g_i on compact subsets of Δ .

Proof. Without loss of generality we may suppose the $f_n(0)$ accumulate somewhere, so say $1 = [1, \dots, 1]$ in an appropriate projective coordinate system consistent with 1.2.2 as already employed in 1.2.3. Consequently if $X_i \in \Gamma(\mathbb{P}^m, H)$

is the equation of a coordinate hyperplane then modulo subsequencing for any $r \in (0, 1)$, we can suppose that for each i , the $f_n^{-1}(X_i) \cap \Delta(r)$, counted with multiplicity are a convergent sequence of 0 cycles in $\bar{\Delta}(r)$ which, after slightly increasing r if necessary, belong to $\bar{\Delta}(s)$ for some $s < r$, which we write as,

$$f_n^{-1}(X_i) = \sum_{z \in \bar{\Delta}(s)} a_{ni}(z)[z] \rightarrow \sum_{z \in \bar{\Delta}(s)} a_i(z)[z]$$

where as in the proof of 1.2.3, $a_{ni}(z) = 0$ for all $|z| \leq t$, t independent of i and n . The disc being Stein we can write $f_n|_{\Delta(r)}$, for an appropriate $1 > R > r$ as,

$$\left[\prod_{w \in \bar{\Delta}_s} (z - w)^{a_{n0}(w)}, \prod_{w \in \bar{\Delta}_s} (z - w)^{a_{n1}(w)} u_1, \dots, \prod_{w \in \bar{\Delta}_s} (z - w)^{a_{nm}(w)} u_m \right]$$

where the u_1, \dots, u_m are units. Now consider the meromorphic functions g_{ni} defined as $f_n^* X_i / f_n^* X_0$ and apply Jensen's formula, i.e. for $z \in \Delta(r)$,

$$\log(|g_{ni}(z)| \cdot |P_{ni}(z)|) = \int_{|w|=R} \log |g_{ni}(w)| \operatorname{Re} \left\{ \frac{w+z}{w-z} \right\} d^c \log |w|^2$$

where $P_{ni}(z)$ is the Blaschke product,

$$P_{ni}(z) = \prod_{w \in \bar{\Delta}(s)} \left\{ \frac{R^2 - \bar{w}z}{R(z-w)} \right\}^{a_{ni}(w) - a_{n0}(w)}$$

Consequently employing the technique of 1.2.3, or more correctly its proof, to bound the integrals over $|w| = R$ of $\log(1 + |g_{ni}|)$ and $\log \left(1 + \frac{1}{|g_{ni}|} \right)$ independently of n we conclude that for some constant C independent of n , $z \in \Delta(r)$,

$$|u_{ni}|(z) \leq C \cdot \frac{(R^2 - rs)^a}{R^b}$$

for non-negative integers a, b independent of n determined by the signs of $a_i(w) - a_0(w)$, so that the u_{ni} converge on subsequencing and we're done. \square

The lemma does not, imply, however that the f_n converge, *e.g.*

$$f_n : \Delta \rightarrow \mathbb{P}^1 : z \rightarrow \left[z - \zeta, z - \zeta + \frac{1}{n} \right]$$

for any $\zeta \in \Delta \setminus 0$, and quite generally, a similar problem presents itself at any point in the common zero locus of the limit g_0, \dots, g_m . On the other hand the maps f_n manifestly converge, and constitute a particularly simple case of Gromov convergence. Deducing this from 1.3.1 is postponed till it is necessary, 2.1.3, and for the moment we summarise a more general setting, thus concentrating on the differences arising from using 1.4 rather than 1.3, beginning with:

Definition 1.3.2. A disc with bubbles Δ^b is a connected 1-dimensional analytic curve with singularities at worst nodes exactly one of whose components is the unit disc Δ , and the closure of every connected component R_z , $z \in R(\Delta^b) := \Delta \cap \text{sing}(\Delta^b)$, of the complement of Δ is a tree of smooth rational curves.

Clearly for τ an integrable $(1, 1)$ form on each component of some Δ^b without a bubble at the origin we can extend 1.8, by way of,

$$\int_{\Delta_r^b} \tau := \int_{\Delta(r)} \tau + \sum_{\substack{z \in R(\Delta^b) \\ |z| < r}} \log \frac{r}{|z|} \int_{R_z} \tau \quad (1.15)$$

and similarly 1.5, which necessarily has similar positivity properties for intersections with effective divisors to the intersection product over discs. In addition we can define a graph Γ_f as,

$$\Gamma_f = (\text{id} \times f)(\Delta) \cup \bigcup_{z \in R(\Delta^b)} z \times f(R_z) \subset \Delta \times X.$$

and we have,

Fact 1.3.3. [Gr] Let X be a complex space admitting a Kähler form ω (or indeed champ admitting the same) then if X is compact any sequence of maps $f_n : \Delta \rightarrow X$ such that, we have a uniform bound in the Nevanlinna area,

$$\int_{\Delta} f_n^* \omega$$

admits a subsequence converging to a disc $f : \Delta^b \rightarrow X$ with bubbles, *i.e.* the graphs Γ_{f_n} of compact sets converge in the Gromov-Hausdorff metric to Γ_f on compact sets. Unlike the corresponding proposition for,

$$\int_{\Delta} f_n^* \omega$$

the converse is, in general, false, to wit: there may be discs f_n converging to a disc with bubbles such that,

$$\int_{\Delta(r)} f_n^* \omega$$

is unbounded for every $r > 0$. Indeed, by 1.15, the Nevanlinna area is bounded in n iff f_n converge to a disc with bubbles without a bubble in the origin, equivalently, the f_n converge normally in a neighbourhood of the origin.

Alternatively, and by way of notation,

Definition 1.3.4. Let X be as in 1.3.3, then the space of maps $\text{Hom}(\Delta, X)$ from the unit disc, may, whenever X is compact, be minimally compactified by the space $\overline{\text{Hom}}(\Delta, X)$ of discs with bubbles. Or, in the Nevanlinna context pointed discs with no bubbling at the point, and, in any case, we have a strict inclusion,

$$\text{Hom}(\Delta, X) \subset \overline{\text{Hom}}(\Delta, X)$$

if and only if X (say smooth for safety) contains a rational curve.

It is, of course, often convenient to move the point, so for $C : (0, 1) \rightarrow \mathbb{R}$, and $K \subset \text{Aut}(\Delta)$ compact another useful variant is that:

$$\left\{ f \in \overline{\text{Hom}}(\Delta, X) : \oint_{\bar{\Delta}(r)} \alpha^* f^* \omega \leq C(r), \alpha \in K \right\} \quad (1.16)$$

is compact, so, in particular $f_n \in \overline{\text{Hom}}(\Delta, X)$ has a convergent subsequence iff for some subsequence f_k , radii $r_m \rightarrow 1$ and $\alpha_k \in \text{Aut}(\Delta)$ convergent, the degrees of the $\alpha_k^* f_k$ at r_m are bounded independently of k for each m .

Consider the same with boundary, *i.e.* X compact, and an effective divisor:

$$B = \sum_i B_i, \text{ each component } B_i \text{ of which is } \mathbb{Q}\text{-Cartier.} \quad (1.17)$$

where the key point is a lemma of Mark Green, *viz*:

Lemma 1.3.5. (*[G]*) *Let (X, B) be as in 1.17, and $f_n : Y \rightarrow X \setminus B$ be a sequence of maps converging uniformly on compact subsets of a Riemann surface to $f : Y \rightarrow X$, then either, f maps to $X \setminus B$ or B , so by induction, there is a minimal (possibly empty) set of components I such that f maps to $B_I := \cap_{i \in I} B_i \setminus \cup_{j \notin I} B_j$.*

Thus if we understand by $\overline{\text{Hom}}(\Delta, X \setminus B)$, the closure of $\text{Hom}(\Delta, X \setminus B)$ in $\overline{\text{Hom}}(\Delta, X)$, we have:

Fact 1.3.6. Let things be as in 1.17, and suppose for every set of components I , $\cap_{i \in I} B_i \setminus \cup_{j \notin I} B_j$ contains no \mathbb{A}^1 's, *i.e.* the Zariski closure in $\cap_{i \in I} B_i$ is \mathbb{P}^1 and it's normalisation meets $\sum_{j \notin I} B_j$ in at most a point, then we have an identity,

$$\{\text{Closure of } \text{Hom}(\Delta, X \setminus B) \text{ in } \text{Hom}(\Delta, X)\} = \overline{\text{Hom}}(\Delta, X \setminus B)$$

and, conversely, if X is smooth with simple normal crossing boundary and identity holds then no stratum B_I contains an \mathbb{A}^1 .

Proof. The converse is for comparison with 1.3.4, and is purely illustrative, so we ignore it. Otherwise, we have a sequence $f_n : \Delta \rightarrow X$ with area uniformly bounded on compact sets, but not converging uniformly. As such there is some $\zeta \in \Delta$ and maps $\varphi_n : \Delta_{R_n} \rightarrow \{z \mid |z - \zeta| < \varepsilon_n\}$ with $R_n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ such that $f_n \circ \varphi_n$ converges uniformly on compact subsets to a map $f : \mathbb{C} \rightarrow X$ (cf. [S]), of bounded area, so its closure is certainly a \mathbb{P}^1 , while the f_n are maps to $X \setminus B$ so 1.3.5 applies to $f_n \circ \varphi_n$. \square

By way of a variation on a theme let us introduce:

Definition 1.3.7. *Let (X, B) be as in 1.17, with $f_n : \Delta \rightarrow X \setminus B$ converging to $f : \Delta^b \rightarrow X$ as per 1.3.2, then we say that bubbles cannot form in the boundary if for every z , $f(R_z)$ meets B in at most $f(z)$, and this only if $f(\Delta^b) \subset B$.*

A close to optimal criteria for which is,

Lemma 1.3.8. *Suppose for $I \neq \emptyset$ the strata B_I do not contain \mathbb{A}^1 's then bubbles cannot form in the boundary.*

Proof. We form the tree, T , whose vertices are the components of Δ^b around z with vertices the intersections between them, and root the tree in the disc component. We aim to prove by decreasing induction on distance (in the graph metric) from the root that no component meets B , except possibly in the point that corresponds to the vertex through which there is the unique path to the root, and this only if the disc component maps to B under f and every vertex in the path from it to the component is contracted to $f(z)$.

To this end, let $R_n \ni z$ be a small disc such that $V_n = f_n(R_n) \subset X \setminus B$ converge to $V = f(R_z)$. Now, quite generally, let us suppose that for some $x \in R_z$, $f(x)$ belongs to some component B_j . Should $x \notin \text{sing}(\Delta^b)$, then $V_n \rightarrow V$ uniformly close to $f(x)$, and one argues as in 1.3.5, *i.e.* if the component of $R_z \ni x$ is not contained in B_j then the intersection of B_j with $f(R_z)$ at $f(x)$ is the limit of that with R_n close to $f(x)$. This works more generally, to wit: if for $p \in V$, c_1, \dots, c_m are the components of V through p , then $p \in B_j$ iff some $c_k \subset B_j$, since: otherwise, there is a tubular neighbourhood $U_j \supset B_j$ such that,

$$V_n \cap U_j \rightarrow V \cap U_j \in H^{2d-2}(U_j)$$

for d the ambient dimension, and the (nil) intersection number is conserved.

Now consider the graph, T_b obtained by colouring vertices contracted to points by f black, white otherwise. Thus with the possible exception of the root, all vertices of valency 1, so, in particular those at maximal distance from the root, may be supposed white. Observe that it's sufficient to perform the induction for white vertices, since otherwise a connected component of blacks at greater distance mapping to a point $p \in B_j$ would force some white vertex to lie in B_j by the above considerations of local intersection numbers. As such, suppose we have a white vertex v such that every white vertex at a greater distance can at most meet B in the edge of its unique path to v , and this only if the path is black. At this point suppose the set I of boundary components containing the vertex is non-empty, then for $j \notin I$, as above, excepting the edge leading to the root, no point of the tree from v on, *i.e.* at greater distance, including the black vertices, can meet B_j . Consequently the vertex v gives an \mathbb{A}^1 in the stratum B_I , contrary to our hypothesis. \square

To which let us add some definitions reflecting the different possibilities that may occur in 1.3.3, 1.3.6, and, 1.3.8,

Definition 1.3.9. *Let Z be a proper (and implicitly without generic points in B) sub-variety of a complex space X , then:*

- (a) *If a sequence of maps $f_n : \Delta \rightarrow X$ of which a subsequence doesn't converge in $\overline{\text{Hom}}(\Delta, X)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of Z , then we say that X is hyperbolic modulo Z .*
- (b) *If a sequence of maps $f_n : \Delta \rightarrow X \setminus B$ of which a subsequence doesn't converge in $\overline{\text{Hom}}(\Delta, X \setminus B)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of $Z \cup B$, then we say that (X, B) is hyperbolic modulo Z .*

- (c) Everything as in (b), except that the f_n are arbitrarily close to Z , then we say that (X, B) is complete hyperbolic modulo Z , thus both this item and (b) encompass (a) for $B = \emptyset$.
- (d) This is defined as the following **infinito** property for (X, B) : every holomorphic map $f : \mathbb{C} \rightarrow X \setminus B$, factors through Z .

Now we can start to bring some order to the discussion,

Fact 1.3.10. Suppose the Bloch principle fails, *i.e.* 1.3.9 (d) does not imply 1.3.9 (b) or (c) as appropriate (*i.e.* for the moment either is permitted, and for surfaces, 2.1.2, we'll give necessary and sufficient algebraic criteria) and, 1.3.7, that bubbles do not form in the boundary, *e.g.* 1.3.8 holds, then there is a sequence $f_n : \Delta \rightarrow X \setminus B$ of discs, unbounded according to the Nevanlinna alternative 1.7, from some fixed radius on, such that:

- (a) The origins $f_n(0)$ are bounded away from $Z \cup B$ if 1.3.9.(b) fails, respectively Z if 1.3.9.(c) fails.
- (b) For $r \in (0, 1)$ outwith a set of finite hyperbolic measure, and possibly after subsequencing, any (weak) accumulation point T of the $T_n(r)$ in 1.5 is supported on $Z \cup B$ - actually $Z \cup W$ for $W \subset B$ the Zariski closure of the union of \mathbb{C} 's (not just the excluded \mathbb{A}^1 's) in boundary strata à la 1.3.6.
- (c) If furthermore X is projective, we may suppose, 1.2.4, that T is closed, and, of course, by (a) and 1.1.1, $D \cdot T \geq 0$, for every effective \mathbb{Q} -Cartier divisor, D , supported in $Z \cup B$, should 1.3.9.(b) fail, respectively Z , if 1.3.9.(c) fails.

Proof. The fact that we can choose the origins $f_n(0)$ as in (a) follows from 1.16, while (c) is just a re-statement of things that have already been proved. As such, if T is the resulting current, the new statement is (b), and we require to prove that T has no mass off Z or $Z \cup B$ as appropriate. To this end, one takes the U of 1.13 to be either one of a relatively compact sequence exhausting $X \setminus (Z \cup B)$, respectively $X \setminus Z$, or better replace \mathbb{I}_U in op. cit. by a continuous $[0, 1]$ valued function vanishing on exactly $Z \cup B$, respectively Z . By 1.2.10, with notation as therein, as soon as $d\nu^U$ has support on $(0, r]$ we find an Ahlfors' current, A , with mass on U and so by the main theorem of [D] there is a \mathbb{C} with bubbles in X meeting U which arises as a limit of discs mapping to $X \setminus B$. By hypothesis bubbles cannot form in the boundary, so, either the \mathbb{C} goes into the boundary and the bubbles lie in U contradicting 1.3.9.(d), or the entire limit lies in U contradicting 1.3.9.(d) again.

We may, therefore, put ourselves in the situation of 1.2.10, and suppose that $d\nu^U$ is supported uniquely in the origin, with, as per the proof of 1.2.10, the limit ϕ of the ratios, ϕ_n , of 1.13, converging to zero Lebesgue almost everywhere. Equally, we may suppose that the areas $a_n(t)$ of every compact $t > 0$ are unbounded in n , since otherwise, by 1.15 the mass is uniquely attributable to a bubble at the origin, which must belong to Z or some boundary strata by

1.3.9.(d), and 1.3.6, but equally cannot contribute mass in U since $f_n(0)$ is bounded away from the bubbling locus.

As such, given $\alpha > 0$, for n sufficiently large, there is some $\varepsilon_n(\alpha)$, such that, $a_n(\varepsilon_n(\alpha)) = \alpha$. Now consider for each α ,

$$\limsup_n \left(\oint_{\Delta(r)} f_n^* \omega \right)^{-1} \int_0^{\varepsilon_n(\alpha)} \phi_n(t) a_n(t) \frac{dt}{t}$$

and suppose for some $\alpha > 0$ this is non-zero. However, for every $x \in (0, 1]$,

$$\int_0^{\varepsilon_n(\alpha)} \phi_n(t) a_n(t) \frac{dt}{t} = \int_0^{x\varepsilon_n(\alpha)} \phi_n(t) a_n(t) \frac{dt}{t} + \int_{x\varepsilon_n(\alpha)}^{\varepsilon_n(\alpha)} \phi_n(t) a_n(t) \frac{dt}{t}$$

and the latter integral is at most $\alpha | \log |x| |$, so normalising the discs $\Delta_{\varepsilon_n(\alpha)}$ to radius 1, every compact Δ_x converges to a disc with bubbles with mass in U , so by 1.15, without loss of generality a non-trivial bubble uniquely at the origin. We may, however, suppose that U is relatively compact in the complement of Z , so such a thing is impossible.

Before we can profit from the above discussion, observe that if $E \subset (0, 1)$ is any set of finite $t^{-1}dt$ measure, then by Ergoff's theorem the convergence of,

$$\int_s^r \mathbb{I}_E \phi_n(t) \frac{dt}{t} \rightarrow \int_s^r \mathbb{I}_E \phi(t) \frac{dt}{t} = 0$$

is uniform in $s \in [0, r]$. Whence if E' is the complement of such an E , $\delta > 0$, and α are given, we may suppose that:

$$\limsup_n \left(\oint_{\Delta(r)} f_n^* \omega \right)^{-1} \int_{\frac{\varepsilon_n(\alpha)}{\delta}}^r \left[\frac{\phi_n(t\delta) a_n(t\delta)}{a_n(t)} \right] \mathbb{I}_{E'}(t\delta) a_n(t) \frac{dt}{t}$$

is some $2\eta \in (0, 1)$ which is independent of δ, α , and E . In particular, for each α there are discs of radius $t_n \in (0, \delta r) \setminus E$ of area at least α and area in U at least $\eta\alpha$. Passing to a sub-sequence indexed by $n = n(\alpha)$ with α growing rapidly, we can as in 1.2.5 choose E a priori, so that:

$$ta'_n(t) \leq (a_n)^{1+\epsilon}, \quad \forall t \in (\varepsilon_n(\alpha), \delta r) \setminus E$$

for some fixed $\epsilon \in (0, 1)$, and since the length of the boundary is bounded by the square root of the left hand side, the discs Δ_{t_n} again yield an Ahlfors current with mass in U , and we conclude once more by [D]. \square

1.4 Tautological Inequality

An appropriate level of generality that covers both 1.1 and 1.2.6 is simply to replace the Dirac delta at p , by a finite sum $\delta_P := \sum_i w_i \delta_{p_i}$, $w_i > 0$, of total mass 1 in 1.1, and we return to the notation of 1.1 for the implied exhaustion of a (regularly bordered) Riemann-surface by g . Thus we have local functions

z_i around p_i such that 1.2 holds for every i on multiplying the right hand side by w_i , while the Euler characteristic in the Nevanlinna sense of 1.1.3 reads,

$$E_\Sigma(r) := \frac{1}{2} \cdot \int_{-\infty}^0 \chi(\Sigma_t \setminus |P|) dt + \#(P) \frac{r}{2} \quad (1.18)$$

for $|P|$ the support of P , and $\#(P) \in \mathbb{N}$ its cardinality. The example 1.1.3 may be generalised as follows:

Fact 1.4.1. Let (X, ω_X) be a complex space, or for that matter analytic champ de Deligne-Mumford, with metric, and \bar{L} the resulting metricisation of the tautological bundle on $\mathbb{P}(\Omega_X^1)$ (EGA notation) then for $f : \Sigma \rightarrow X$, with derivative, $f' : \Sigma \rightarrow \mathbb{P}(\Omega_X^1)$, unramified at P ,

$$\int_{\Sigma_r} f^* c_1(\bar{L}) + E_\Sigma(r) + \text{Ram}_f(r) + \sum_i w_i \log \frac{1}{w_i} \|f_* \frac{\partial}{\partial z_i}\| (p_i) = \int_{\partial \Sigma_r} \log \|\partial\|_{\omega_X} d^c g$$

for z_i as above, and for ram_f the ramification divisor,

$$\text{Ram}_f(r) := \frac{1}{2} \cdot \int_{-\infty}^r \#(\text{ram}_f \cap \Sigma_t)$$

Proof. Define a function $|F|$ by,

$$f^* \omega_X = |F|^2 dg d^c g$$

then $f^* c_1(\bar{L})$ is $dd^c \log |F|^2$ Lebesgue a.e. on Σ and one integrates by parts. \square

Now the right hand side of 1.4.1 certainly admits the bound,

$$\log[(\text{length}_{\omega_X})(\partial \Sigma_r)] \quad (1.19)$$

which if X is compact is wholly negligible in practice by 1.2.5. However, if we pass to a divisorial boundary (X, B) à la 1.17, but X smooth at every point of B with the latter simple normal crossing divisor, so that X of 1.4.1 becomes $X \setminus B$, then the estimability of 1.19 is wholly dependent on how the (Nevanlinna) area computed in $\omega_{X \setminus B}$ compares with that computed in a smooth metric ω_X . For example, if $x_1 \dots x_p = 0$ is a local equation for the boundary and y_j the other coordinates then a metric of the form,

$$\sum_i \frac{dd^c |x_i|^2}{|x_i|^2} + \sum_j dd^c |y_j|^2 \quad (1.20)$$

on $X \setminus B$, affords an area that has no a priori relation with the same on X . On the other hand if we take a complete metric on $X \setminus B$, *i.e.* everywhere locally commensurable to,

$$\sum_i \frac{dd^c |x_i|^2}{|x_i|^2 \log^2 |x_i|^2} + \sum_j dd^c |y_j|^2 \quad (1.21)$$

Then there is a constant $N > 0$ such that,

$$\omega_X \leq \omega_{X \setminus B} \leq N\omega_X - \sum_i dd^c \log \log^2 \|\mathbb{I}_{B_i}\|^2 \quad (1.22)$$

supposing as we may that the sup of any $\|\mathbb{I}_{B_i}\|^2$ is at most e^{-1} . Consequently for any $f : \Sigma \rightarrow X$ which we allow to meet B , but say $P \cap f^{-1}(B)$ for convenience, an integration by parts together with our bound on the distance to the B_i gives,

$$\int_{\Sigma_r} f^* \omega_{X \setminus B} \leq N \int_{\Sigma_r} f^* \omega_X + \log |\log \|\mathbb{I}_{B_i}\|^2| (\delta_P) \quad (1.23)$$

so that controlling the distance of P to B gives control on the area computed in the complete metric, from which, the utility of:

Fact 1.4.2. Let (X, B) be compact with smooth simple normal crossing boundary B , and L^c a metricisation of the tautological bundle of $\mathbb{P}(\Omega_X^1(\log B))$ resulting from a complete metric $\omega_{X \setminus B}$ on $X \setminus B$, then for any $f : \Sigma \rightarrow X$ with logarithmic derivative, $f' : \Sigma \rightarrow \mathbb{P}(\Omega_X^1(\log B))$, unramified at P , and $f(P)$ missing B ,

$$\begin{aligned} \int_{\Sigma_r} f^* c_1(L^c) + E_\Sigma(r) + \text{Ram}_f^B(r) + \sum_i w_i \log \left[\frac{1}{w_i} \|f_* \frac{\partial}{\partial z_i}\|_{\omega_{X \setminus B}} \right] (p_i) = \\ \text{Rad}_f^B(r) + \int_{\partial \Sigma_r} \log \|\partial\|_{\omega_X} d^c g \end{aligned}$$

for z_i , $\text{Ram}_f^B(r)$ as in 1.4.1, except that one integrates the part ram_f^B of the ramification divisor which is not supported in B , while for rad_f^B the divisor $f^{-1}(B)$ counted without multiplicity,

$$\text{Rad}_f^B(r) := \frac{1}{2} \cdot \int_{-\infty}^r \# \left(\text{rad}_f^B \cap \Sigma_t \right)$$

Proof. Exactly as per 1.4.1. □

While 1.4.2 together with the estimate 1.23 is the most accurate reflection of the geometry, it's not exactly admissible to apply something like 1.2.4 to the derivatives with the complete metric rather than a smooth metric, \overline{L} . Fortunately, a comparison is quite easy, since:

$$c_1(\overline{L}) = c_1(L^c) + dd^c(\psi), \text{ where, } 0 \leq \psi \leq \log \log^2 \|\mathbb{I}_B\|^2 \quad (1.24)$$

and so we obtain,

Fact 1.4.3. Let everything be as in 1.4.2 with \overline{L} as above, then for a constant C depending only on the metrics:

$$-\log |\log \|\mathbb{I}_B\|^2| (\delta_P) \leq \int_{\Sigma_r} (c_1(\overline{L}) - c_1(L^c)) \leq \log [|\log \|\mathbb{I}_B\|^2| (\delta_P) + C \int_{\Sigma_r} \omega_X]$$

Proof. One integrates by parts, and the lower bound is the easier one, with estimation exactly as in 1.23, while an upper bound is,

$$\int_{\partial\Sigma_r} \log |\log \|\mathbb{I}_B\|^2| d^c g \leq \log \int_{\partial\Sigma_r} |\log \|\mathbb{I}_B\|^2| d^c g$$

and one concludes by 1.1.1. \square

We conclude the pre-liminaries, by observing that whether in 1.4.1 or 1.4.2 we require some control on the derivative at the origin, which we do by:

Lemma 1.4.4. *Suppose a sequence $\{f_n\} \in \text{Hom}(\Delta, X)$ is given which is not eventually arbitrarily close (in the compact open sense) to a Zariski closed subset Z of X , and that no subsequence of $\{f_n\}$ converges then after subsequencing there is a convergent sequence $\alpha_n \in \text{Aut}(\Delta)$ such that $\text{dist}(\alpha_n^* f_n(0), Z)$ and $\|(\alpha_n)_* (f_n)_* (\frac{\partial}{\partial z})\|_{\omega_X}(0)$ are bounded away from zero independently of n .*

Proof. By hypothesis there is a disc of radius $r < 1$, with $f_n(\overline{\Delta}_r) \not\subset U$ for U some open neighbourhood of Z , and $|df_n|_{\text{euc}}$ large somewhere on $\overline{\Delta}_r$. Provided we simply take automorphisms that move the origin to somewhere in $\overline{\Delta}_r$ these will converge after subsequencing, so we might as well say that $|df_n|_{\text{euc}}(0)$ is as large as we like. So consider a connected component U'_n of points in Δ_r which are a suitable distance ε away from Z , which contains an open subset of points U''_n of points a distance 2ε away. Now choose $u_n \in U''_n$, and consider the ray $R \ni u_n$ of constant argument joining u_n to the origin. Without loss of generality, $0 \notin U'_n$, so the connected component, γ , of $U'_n \cap R$ containing u_n has some boundary (going towards the origin) at v_n with $\text{dist}(f_n(v_n), Z) = \varepsilon$. In particular $\text{dist}(f_n(u_n), f_n(v_n)) > \varepsilon$, so:

$$\varepsilon \leq \int_{\gamma} \| (f_n)_* (\frac{\partial}{\partial z}) \|_{\omega_X} |dz|$$

and we find points in $\gamma (\subset U'_n$ by definition) of derivative at least ε/r . \square

2 Algebraic surfaces

2.1 Minimal models

The log-minimal model programme for quasi-projective surfaces will afford optimal conditions for avoiding bubbles in the boundary as required by 1.3.10. Starting from a smooth projective surface S with simple normal crossing boundary B , one runs the minimal model programme for $K_S + B$. The result $\rho : (S, B) \rightarrow (S', B')$ either has $K_{S'} + B'$ nef, or $S \setminus B$ is covered, [KM], by \mathbb{A}^1 's, and is the opposite of what we're interested in. At the initial stage in the programme any -1 curve in S curve that does not meet B can always be contracted, so, we may simplify the discussion by supposing that there are no such. As a result an initial move in the programme must involve a curve that meets the boundary,

and it always reduces the obstruction to bubbling in the boundary. Such curves need not, however, be contained in the boundary, so $\rho^{-1}(B')$ can be strictly bigger than B . Further, our notation, 1.17 only ever permits divisorial boundaries, and it can perfectly well happen that a connected component of B is contracted to a point. In this latter case the correct thing to do is to fill the point, and omit it from the boundary. Indeed if, for example, B were a -1 curve so that S' would be the contraction of the same to a point p , then by [CHO] discs converge in $S' \setminus p$ iff they converge in S' , but a property such as 1.3.9.(d) for $S' \setminus p$ does not imply the same for S' . The behaviour of discs in the general case of a boundary component contracting to a point is wholly identical up to the action of a finite group, *i.e.* the resulting point $p \in S'$ is an isolated quotient singularity, and discs converge in $S' \setminus p$ iff they converge in S'_p , where $S'_p \rightarrow S'$ is the minimal champ de Deligne-Mumford over S' which is smooth at p , or, in a perhaps more familiar language, the filling at p is by way of an orbifold point.

On the other hand, (S', B') , or better a yet smaller object, the canonical model, is where the natural geometry of the pair (S, B) is to be found, *e.g.* Kähler-Einstein metrics are metrics on the canonical model, and whence on (S, B) reflect both the above phenomenon, *i.e.* they are not complete at components of B which are contracted to quotient singularities, but they will be complete around certain divisors in the interior $S \setminus B$ whenever $\rho^{-1}(B')$ is strictly bigger than B . As a result, hyperbolicity questions about smooth projective surfaces S with simple normal crossing divisor B are somewhat ill posed. Nevertheless, the canonical model, naturality notwithstanding, may not be right for the hyperbolicity problem that one wishes to understand, and so we make:

Fact/Defintion 2.1.1. In the first place if X is a normal variety, or even just complex space, with quotient singularities, then there is a unique, [V], champ de Deligne-Mumford, \mathcal{X} , the Vistoli covering champ, on X (*i.e.* with the same moduli) such that \mathcal{X} is smooth and $\mathcal{X} \rightarrow X$ almost étale. Thus, if (S', B') is the minimal model of some quasi-projective algebraic surface (S, B) , there is such a champ $S'' \rightarrow S'$, a general pair (S'', B'') with log-canonical singularities may, however, have an isolated set $P \subset S''$ of elliptic Gorenstein singularities (which can only occur on contracting a connected component of B' which is numerically elliptic) so there is only a champ S'' on S'' equal to the Vistoli covering champ (whence smooth) over $S'' \setminus P$, and an isomorphism around P otherwise. In the case that (S'', B'') is the canonical model, we also distinguish an intermediate object $(S', B') \rightarrow (S_0, B_0) \rightarrow (S'', B'')$, which will be said to have a canonical boundary, if it is obtained from S' by modifying in the boundary alone, and every generic point of B_0 and B'' coincide, so, in particular it has the same set, P , of elliptic Gorenstein as S'' , and so there is a champ $S_0 \rightarrow S_0$ with the same properties. In all cases the boundaries, B' , B'' , and B_0 in the respective champ are, étale locally, simple normal crossing, so, inter alia $P \cap B_0 = \emptyset$.

The relation of the above with the metric structure has largely already been described since the singularities of S'' remain isolated quotient except in the elliptic Gorenstein case. Here, the Kähler-Einstein metric is complete in the complement of P , and, indeed, modelled as in 1.21 around the fibre B'_p . Even

locally, around P , however, B'_P is an obstruction to completeness in the sense of 1.3.9.(c), albeit never an occasion of bubbling in the boundary, cf. 1.3.8, so that we can only expect such completeness in S_0 or S'' . Let us therefore make appropriate changes to 1.3.9 to reflect this situation,

Summary/Defintion 2.1.2. (1.3.9.bis) If X of 1.3.9 is smooth then it matters not a jot to any of properties (a)-(c) as to whether the maps f_n take values in X or omit a co-dimension 2 subset, and, idem, if X were a smooth champ. As such if (S', B') is the minimal model of a smooth quasi-projective surface $S \setminus B$, or is the model (S_0, B_0) with canonical boundary, then for Z as per op. cit.:

(b) If a sequence of maps $f_n : \Delta \rightarrow S' \setminus B'$ of which a sequence doesn't converge in $\overline{\text{Hom}}(\Delta, S' \setminus B') \subseteq \overline{\text{Hom}}(\Delta, S')$ is (compact open sense) contained in arbitrarily small neighbourhoods of $Z \cup B'$, then we say that the minimal model of the smooth quasi-projective surface $S \setminus B$, or (S', B') if this is simply a pair with klt singularities, and there's no danger of confusion, is hyperbolic modulo Z .

In addition, a canonical boundary is a necessary condition for completeness. Indeed, if, say, $\mathcal{B}_i \subset \mathcal{B}'$ were the champ over a boundary component B_i , the Euler characteristic of $\mathcal{B}_i \setminus \{\text{sing}(\mathcal{B})\}$ is $-(K_{S'} + B') \cdot B_i$. As such if this is zero, the universal cover of $\mathcal{B}_i \setminus \{\text{sing}(\mathcal{B})\}$ is \mathbb{C} , so arbitrarily small neighbourhoods of such contains discs that are as big as one pleases. Whence,

(c) If a sequence of maps $f_n : \Delta \rightarrow S_0 \setminus \{B_0 \cup P\}$, for P the (possibly empty, always isolated) elliptic Gorenstein locus of which a sequence doesn't converge in $\overline{\text{Hom}}(\Delta, S_0 \setminus \{B_0 \cup P\}) \subseteq \overline{\text{Hom}}(\Delta, S_0)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of Z , then we say that the model of the smooth quasi-projective surface $S \setminus B$ with canonical boundary, or (S_0, B_0) if this is simply a pair with log-canonical singularities, and, there is no danger of confusion, is complete hyperbolic modulo Z .

Of course, as above, strict positivity of $K_{S_0} + B_0$ on the boundary is a necessary condition for complete hyperbolicity, but, equally this always holds on the canonical model in the only case where the definitions can be non-empty, viz: $S \setminus B$ of general type, while, finally, 1.3.9.(d) is changed to,

(d) If every holomorphic map, $f : \mathbb{C} \rightarrow S' \setminus B'$ to the covering champ of 2.1.1, respectively $S_0 \setminus \{B_0 \cup P\}$, factors through Z then we say that **infinito** or GG (Green-Griffiths) holds for the minimal model, respectively model with canonical boundary, of the smooth quasi-projective surface $S \setminus B$, or even just that (GG) holds for (S', B') , respectively (S_0, B_0) , if this is simply a pair with klt, respectively canonical, singularities, and there is no danger of confusion. So, trivially, (GG) always follows from (b) or (c) as appropriate.

Modulo an issue about components of the boundary that may have nodes, the above considerations on the Euler characteristic show that both \mathcal{B}' and \mathcal{B}_0 have étale neighbourhoods in which the strata, in the sense of 1.3.5, do not contain \mathbb{A}^1 's. Unfortunately, there's no reason for this étale covering to extend over S' , so 1.3.8 does not apply to conclude no bubbling in the boundary. Nevertheless,

Proposition 2.1.3. *Let (S', B') be a minimal model of a quasi-projective surface $S \setminus B$, and $f_n : \Delta \rightarrow S' \setminus B'$ (or, better, $\Delta \rightarrow S' \setminus \{B' \cup \text{sing}(S')\}$), since,*

it's easier, and, as we've observed the result is the same) converge to a disc with bubbles in S' (albeit here non-schematic points in the interior cannot be omitted), then, 1.3.7, the bubbles do not form in the boundary.

Proof. We use 1.3.1, and the notation therein, so $f_n = [f_{n0}, \dots, f_{nm}] \in \mathbb{P}^m$ with f_{ni} converging uniformly on compacts to some g_i . As such if a bubble forms about $\zeta \in \Delta$ it's uniquely because all the g_i vanish in ζ , to some order $p_i > 0$. Replacing $\Delta \ni \zeta$ by a small disc centred on the origin, and subsequencing as necessary, we can say that f_{ni} vanishes at points $t_{nij} \rightarrow 0$, $1 \leq j \leq J_i$, to order q_{ij} with J_i , q_{ij} independent of n , and, of course $p_i = \sum_j q_{ij}$. So, in projective coordinates, close to the bubble:

$$f_n : z \mapsto \left[\left(\prod_{j \in J_0} (z + c_{n0j})^{q_{0j}} \right) u_{n0}(z), \dots, \left(\prod_{j \in J_m} (z + c_{nmj})^{q_{mj}} \right) u_{nm}(z) \right]$$

with u_{ni} units converging to units u_i , and we may as well suppose $u_i(0) = 1$. As such we can write,

$$f_n(z) = [P_0(z, t_{n0})u_0(z), \dots, P_m(z, t_{nm})u_m(z)]^{\alpha_n(z)}$$

For some $\alpha_n(z)$ in PGL_{m+1} identified with a diagonal matrix in the image of the exponential from \mathfrak{pgl}_{m+1} , *i.e.* close to the identity, and $P_i(z, t_{ni})$ a close to monic polynomial of degree p_i . The images of the $P_i(z, t_{ni})$ in the space of such polynomials, have, for each $N \in \mathbb{N}$ a Zariski closure of the set of such with $n \geq N$, which by Noetherian induction must stabilise, and it is in this sense that we understand the Zariski closure T in the space of polynomials.

From which, it's almost automatic that the f_n converge in \mathbb{P}^m with at worst bubbles, *i.e.* up to a small perturbation our sequence belongs to one of finitely many Hilbert schemes compactifying rational maps to \mathbb{P}^m . Our proposition, however, regards S' , which for ease of notation we'll suppose equal to S , so we need a similar sort of interpolation but in S rather than just \mathbb{P}^m . To this end, suppose that some generic projection $p : S \rightarrow \mathbb{P}^2$ has been chosen a priori, so, without loss of generality the limiting (small) disc with bubbles only meets the ramification, R , in smooth points, and, again keeping the discs around the point of bubbling sufficiently small, we can suppose that,

$$(f_n)^* R = \sum_{k \in K} e_k [c_{nk}]$$

are supported in some compact with e_k , K independent of n . In addition there is some N , independent of n , such that if some map g agrees with $p \circ f_n$ to order N at some c_{nk} then around the same g lifts to S with the same order around R . Now we do what we did before for the initial polynomial, but this time for the automorphisms α_n , which we identify with a vector of functions h_{ni} vanishing at the origin, and converging to zero uniformly. At each c_{nk} these have a Taylor expansion to order N , giving a projection a_n of each α_n to (a bounded subset) of \mathbb{A}^{NK} . Again, in the above \liminf sense we take the Zariski closure of our

sequence to get some space of automorphisms $\alpha(z, a) \in A \subset \text{Hom}(\Delta, \text{PGL}_3)$, which are just exponentials of diagonal polynomial matrices, and which satisfy,

$$\alpha(z, a_n) = \alpha_n(z) \bmod \mathfrak{m}_{c_{nk}}^N, \quad \forall 1 \leq k \leq K$$

Identifying A with the subset of polynomials affording it, and now taking the Zariski closure, V , of our sequence $v_n = (a_n, t_n) \in A \times T$, we have maps,

$$f(z, a, t) : \Delta \times V (\subset A \times T) \longrightarrow \mathbb{P}^2 : z \mapsto [P_0(z, t)u_0(z), \dots, P_2(z, t)u_2(z)]^{\alpha(a, z)}$$

with $f_n(z) = f(z, v)^{\beta_n(z)}$, and $\beta_n(z) \in \text{PGL}_3$, close to \mathbb{I} for all z and identically \mathbb{I} to high order whenever $z \in (f_n)^{-1}(R)$. As such the $f(z, v_n)$ lift to S , and lifting is a Zariski closed condition, so we have a lifting $F(z, v) : \Delta \times V \rightarrow S$ of $f(z, v)$. Of course, no f_n need equal any $F(z, v_n)$, but the distance between their graphs goes to zero at points of bubbling, so, recalling $S' = S$ for ease of notation, it will suffice to answer the question for some meromorphic mapping,

$$F : X := \Delta \times \Delta : (z, t) \mapsto f_t(z) \quad (\in S \setminus \{B \cup \text{sing}(S)\}, \text{ if } t \neq 0)$$

from a bi-disc, defined everywhere except the origin, with bubbles forming as $t \rightarrow 0$. Put t to be the 2nd projection, and consider a resolution ϕ of F ,

$$\begin{array}{ccccccc} \Delta & \xleftarrow[t]{} & X & \xleftarrow[\pi]{} & \tilde{X} & \xleftarrow[\tilde{\pi}]{} & \mathcal{X} \\ & & & & \downarrow \phi & & \downarrow \tilde{\phi} \\ & & & & S & \xleftarrow{} & \mathcal{S} \end{array}$$

Where $\tilde{\phi}$, and $\tilde{\pi}$ are defined by normalising the dominant component of $\tilde{X} \times_S \mathcal{S}$. As such, if we suppose, as we may, that π is a minimal resolution of F obtained by blowing up in closed points, then $\mathcal{X} \rightarrow \tilde{X}$ has pure ramification, and this only over components, c , of $(t\pi)^{-1}(0)$ which are contracted to points in the singularities of S . Consequently, at the price of replacing \mathcal{X} by an almost étale bi-rational cover, we may, [SGAI, XIII,5.3], suppose that \mathcal{X} is smooth, and:

$$K_{\mathcal{X}} = \tilde{\pi}^* K_{\tilde{X}} + \sum_c (1 - \frac{1}{n_c})c$$

for some $n_c \in \mathbb{N}$. Similarly $(\mathcal{S}, \mathcal{B})$ has simple normal crossings so for $\{b\}$ the components of $\phi^{-1}(B)$, and, implicitly, omitting c if it already appears among the b 's,

$$K_{\tilde{X}} + \sum_b b + \sum_c (1 - \frac{1}{n_c})c = \phi^*(K_S + B) + F + G$$

for F and G (possibly nil) effective \mathbb{Q} -divisors with the former supported in the fibre over $t\pi$, and the latter transverse to it. Among the b 's and c 's there may or may not be the proper transform $\tilde{\Delta}$ of the central fibre, Δ , of t , but were it amongst the b 's we put $a = 1$, should it be among the c 's (so, implicitly not a b) we put $a = 1 - 1/n_c$, for the appropriate c , and 0 otherwise. In all cases,

$$K_{\tilde{X}} + a\tilde{\Delta} = \pi^*(K_X + a\Delta) + D_0$$

for D_0 an effective divisor contracted by π , so that for,

$$D = D_0 + \sum_{b \neq \tilde{\Delta}} b + \sum_{c \neq \tilde{\Delta}} (1 - \frac{1}{n_c})c$$

and e any divisor contracted by π , $K_S + B$ nef. yields:

$$(D - F) \cdot e \geq 0$$

so, as ever, the negativity of the intersection form forces $F = D + E$ for E effective, possibly nil, and contracted by π . Now suppose there is a bubble in the boundary, then it must be one of the above b 's, and F can have no support on it, contrary to the definition of D . \square

As such we may refine 1.3.10 for surfaces by way of,

Fact 2.1.4. Let (S, B) , with log-canonical singularities, be a model of a quasi-projective surface with $K_S + B$ nef., and (\tilde{S}, \tilde{B}) the modification in the elliptic Gorenstein singularities, P , if any, which is isomorphic to the minimal model around a neighbourhood of the same, and (S, B) otherwise, then should the Bloch principle fail, *i.e.* GG, 2.1.2.(d), holds but 2.1.2.(b) or (c) fails (the former, rather than the latter, will be the case of interest iff $K_S + B$ is not strictly positive on the boundary) then there is a closed positive current T on a smooth champ $\tilde{S} \rightarrow \tilde{S}$ over \tilde{S} such that all of 1.3.10.(a)-(b) hold with W of op. cit. exactly the boundary components, if any, on which $K_S + B$ is nil. In particular if $P \neq \emptyset$, any components of T in the fibres of $\tilde{S} \rightarrow S$ are a chimera, *i.e.*, by construction, they can be blown down.

2.2 Proof of the principle

Let X be an algebraic surface, and T a closed positive current in the Nevanlinna sense arising from a sequence of discs $f_n : \Delta \rightarrow X$ at some radius r as in 1.2.4, and suppose, in addition, that there is a simple normal crossing divisor $D \subset S$ such that $T = \mathbb{I}_D T$, or, equivalently, $T = \sum_i \lambda_i D_i$, for D_i the components of D , and $\lambda_i \geq 0$. As in 1.4.2, we let $f'_n : \Delta \rightarrow P := \mathbb{P}(\Omega_X^1(\log D))$ be their logarithmic derivatives, with L the tautological bundle, and since our goal is a lower bound for the (Nevanlinna) degree of $(f'_n)^* c_1(L)$ in terms of the (Nevanlinna) area on X , we may, without loss of generality, suppose:

$$\limsup_n \left(\int_{\Delta(r)} f_n^* \omega_X \right)^{-1} \int_{\Delta(r)} (f'_n)^* c_1(\bar{L}) < \infty \quad (2.1)$$

Consequently as per 1.2.6 we get a well defined derivative T' on P lifting T , and we assert,

Proposition 2.2.1. *If (X, D) is obtained from a pair (Y, B) of a simple normal crossing divisor B on a smooth surface Y by blowing up in the crossings of B , and the origins $f_n(0)$ are bounded away from D , then:*

$$L \cdot T' \geq (K_X + D) \cdot T$$

Proof. For every curve C in the support of D there is a short exact sequence,

$$0 \rightarrow \mathcal{O}_C(K_X + D) \rightarrow \Omega_X(\log D) \otimes \mathcal{O}_C \rightarrow \mathcal{O}_C \rightarrow 0$$

where the last map is the residue map, which in turn defines a section S_C in the fibre P_C of P over C . The S_C are disjoint among themselves, and:

$$\mathcal{O}_{P_C}(S_C) = \mathcal{O}_{P_C}(L - \pi^*(K_X + D))$$

The map $\rho : X \rightarrow Y$ distinguishes curves in D into those arising as proper transforms of curves in B , and those blown down by ρ . For C of the latter type, S_C , and whence L , are nef. on P_C . Now define,

$$T_C := \begin{cases} \mathbb{I}_{P_C} T & C \text{ not contracted by } \rho, \\ \mathbb{I}_{P_C} T - \sum_{C' \in |D| \setminus C} \mathbb{I}_{\pi^{-1}(C \cap C')} T & \text{otherwise} \end{cases}$$

then $T = \sum_C T_C$, and identifying a curve B_i in B with its proper transform,

$$L \cdot T \geq \sum_i L \cdot T_{B_i} \geq (K_X + D) \cdot T + \sum_i S_{B_i} \cdot T_{B_i}$$

where, at a risk of notational confusion with 1.1.1, here and elsewhere $\mathbb{I}_* T$ is the part of T supported on the set $*$, which itself is the push-forward of a closed positive current on $*$ if $*$ is, inter alia, Zariski closed.

We thus require to prove that the $S_{B_i} \cdot T_{B_i} \geq 0$, to which end let's momentarily forget all of the above and make,

Intermission 2.2.2. Let $V \subset W$ be a proper subvariety, and $F \subset V$ a Cartier divisor on V . For each $n \in \mathbb{N}$ let $\pi_n : W_n \rightarrow W$ be the blow up in $\mathcal{O}_V(-nD)$ viewed as a sheaf of ideals on W , with E_n the exceptional divisor. If, for simplicity of exposition, we suppose all of F, V, W smooth with V a divisor, then W_n has only a \mathbb{Z}/n quotient singularity on E_n , and this is disjoint from the proper transform \tilde{V} of V . As such there is a smallest smooth champ $\mathcal{W}_n \rightarrow W_n$ on which currents *etc.* may be understood without repeated use of the preparation theorem.

Given where we started, we obviously suppose there is a current T arising from discs on W by way of 1.2.4, with T_n liftings to \mathcal{W}_n as in 1.2.6, with origins bounded away from F , and we assert:

Claim 2.2.3. *If for every $n \in \mathbb{N}$, $(\pi_n)_*(\mathbb{I}_{E_n \setminus \tilde{V}} T_n) = 0$ then, $F \cdot \mathbb{I}_V T \geq 0$.*

Proof. For each n write:

$$T_n = \mathbb{I}_{\tilde{V}} T_n + \mathbb{I}_{E_n \setminus \tilde{V}} T_n + \mathbb{I}_{\mathcal{W}_n \setminus \pi^{-1}(V)} T$$

then by 1.1.1, $E_n \cdot T_n \geq 0$, while E_n is negative on the fibres of π_n so,

$$E_n \cdot \mathbb{I}_{\tilde{V}} T_n + E_n \mathbb{I}_{\mathcal{W}_n \setminus \pi^{-1}(V)} T \geq 0$$

and the first term is $nF \cdot \mathbb{I}_V T$, while, $\pi_n^* V = \tilde{V} + nE_n$, so the second is bounded above by, $\frac{1}{n} V \cdot \mathbb{I}_{W \setminus V} T$. \square

The intermission over, we see that we'll be done if we can establish the conditions of 2.2.3 in the particular case of $W = P$, $V = P_C$, and $F = S$ for C in the support of our boundary. To this end suppose locally D is given by an equation $x = 0$, and y is the other coordinate. In a neighbourhood of S we have a coordinate $t = \frac{dy}{dx/x}$ (or $\frac{dy/y}{dx/x}$ if we're close to a crossing of D) with respect to which $\pi_n : P_n \rightarrow P$ is a resolution of the ideal $I_n = (t^n, x)$, i.e. $\mathcal{O}_{P_n}(-E_n) = \pi^{-1}(I_n)$, so to be an open neighbourhood a distance ε off the proper transform of P_C amounts to the condition,

$$\frac{|x|}{|x| + |t|^n} \geq \varepsilon \Leftrightarrow |dy| \text{ (resp. } \left| \frac{dy}{y} \right| \text{ near a crossing)} \leq \left(\frac{1 - \varepsilon}{\varepsilon} \right)^{\frac{1}{n}} \frac{|dx|}{|x|^{1-1/n}}.$$

Consequently we need an estimate for the measure, $T \left(\mathbb{I}_{\delta, \varepsilon} \frac{dd^c |x|^2}{|x|^{2-2/n}} \right)$ for $\mathbb{I}_{\delta, \varepsilon}$ a small neighbourhood a distance δ around C , but ε off the proper transform of P_C . This is however rather easy, e.g. by 1.22, the complete metric is absolutely integrable so the above satisfies: $\ll \delta^{2/n} \log^2 \frac{1}{\delta}$. \square

From which we can proceed to our main application:

Corollary 2.2.4. *Let (S, B) , with log-canonical singularities, be a model of a quasi-projective surface with $K_S + B$ nef., and suppose there is a proper subvariety $Z \subset S$ without generic points in the boundary such that the Green-Griffiths property, 2.1.2.(d), holds for (S, B) then $S \setminus B$ is hyperbolic modulo Z , 2.1.2.(b), and it is complete hyperbolic modulo Z , 2.1.2.(c), iff $(K_S + B) \cdot C > 0$ for every curve $C \subset B$.*

Proof. We aim for 2.1.2.(b) first, and suppose it fails. Let, $\rho : X \rightarrow S$ be a smooth model such that the total transforms of B , any elliptic Gorenstein singularities, P , and Z , form a smooth simple normal crossing divisor D , and ρ factors through some $X \rightarrow Y$ as hypothesised in 2.2.1. We divide D into the part ∂ supported on $\pi^{-1}(B \cup P)$, and the rest F . By the definition of log-canonical singularities, there is a divisor $E \geq 0$ contracted by ρ such that,

$$(K_X + \partial) = \rho^*(K_S + B) + E \quad (2.2)$$

By 1.3.10 we can find a closed positive current T on X supported in D afforded by discs f_n at a radius r in the Nevanlinna sense, 1.6, such that the origins $f_n(0)$ are bounded away from D , and by 1.4.4 we may even suppose that the $f'_n(0)$ are uniformly bounded below. Consequently, 1.4.2 and 1.4.3 together with 1.2.5 apply to find a derived current T' on $\mathbb{P}(\Omega_X^1(\log D))$ lifting T such that,

$$L \cdot T' \leq \limsup_n \left(\int_{\Delta(r)} f_n^* \omega_X \right)^{-1} \text{Rad}_{f_n}^F(r)$$

Since the origins are bounded away from D , so, a fortiori F , by 1.1.1, the right hand side is at most $F \cdot T$, so that by 2.2.1, and 2.2,

$$(K_S + B) \cdot \rho_* T \leq (K_X + D) \cdot T - F \cdot T \leq L \cdot T' - F \cdot T \leq 0$$

Since GG holds for (S, B) it must have log-general type and admits a canonical model $\nu : (S, B) \rightarrow (S'', B'')$, so that the support of T is contracted by ν , and $T^2 < 0$. However, all curves contracted by ν necessarily belong to $Z \cup B$, so by virtue of the positions of the $f_n(0)$, $T^2 \geq 0$ by 1.1.1, which is nonsense.

As such, let us turn to the discussion of complete hyperbolicity. The only if has been covered in 2.1.2, and by 2.1.3 together with an appeal to [D] as in 1.3.10 the current T of 2.1.4 cannot be supported in the boundary \tilde{B} of the modification, $\tilde{S} \rightarrow S$, in 2.1.4 except when this differs from S , *i.e.* in the presence of elliptic Gorenstein singularities. Let us clarify this latter, and very minor, difficulty: supposing these exist in S , then $(\tilde{S}, \tilde{B}) \rightarrow (S, B)$ is a modification in the elliptic Gorenstein singularities, P , alone, $K_{\tilde{S}} + \tilde{B}$ is the pull-back of $K_S + B$, the pre-image of every point $p \in P$ is a connected component, \tilde{B}_p of \tilde{B} , and \tilde{S} has at worst quotient singularities. In particular, we have a Vistoli covering champ $\tilde{\mathcal{S}} \rightarrow \tilde{S}$, and if \tilde{Z} is the proper transform of the locus $Z \subset S$ and we arrive to prove that discs are arbitrarily close to $\tilde{Z} + \tilde{B}_P$ in \tilde{S} , then, plainly they're arbitrarily close to Z . As such, the problem of elliptic Gorenstein singularities may be ignored by replacing (S, B) by (\tilde{S}, \tilde{B}) , and every occurrence of close to Z by close to Z and elliptic Gorenstein components of the boundary. A more serious difficulty is that the above upper bound for the intersection with L may fail because of 1.23. To minimise the changes, let's proceed by induction, *viz.*, bearing in mind the above remarks about the elliptic Gorenstein case, prove arbitrary proximity to $Z \cup B'$ for $B' \subset B$ a (eventually empty) divisor on S . We can confine our attention to discs close to a connected component of $Z \cup B'$, and, as in 2.1.1 we take an almost étale map $\mathcal{S} \rightarrow S$ from a champ de Deligne-Mumford in which the boundary \mathcal{B} becomes simple normal crossing. Now fix a component $C \subset B'$, with $\mathcal{C} \rightarrow C$ the champ over it, and put $B = C + C'$, $B' = C + B''$. Away from C we blow up as before with a view to arguing as in 2.2.1. Around C we proceed quite differently. Firstly \mathcal{C} may have nodes. These are disjoint from the rest of B but may intersect Z , if we modify in them (blow up at the \mathcal{S} level and take moduli) we get some exceptional divisor N , together with proper transforms \tilde{C} , \tilde{Z} , of C , respectively Z , and if we can prove that large discs must be arbitrarily close to $\tilde{Z} + N + B''$, then on blowing down the discs are either small because they're close to a node, or they're arbitrarily close to $Z + B''$. As such we may, without loss of generality, suppose that \mathcal{C} is smooth, and for $n \in \mathbb{N}$, we let $p_n : \mathcal{S}_n \rightarrow \mathcal{S}$ be the champ obtained by taking a n th root along \mathcal{C} . Of course Z may be rather far from simple normal crossing in \mathcal{S}_n , so we blow up the champ \mathcal{S}_n to some $\rho : \mathcal{X} \rightarrow \mathcal{S}$ in which the total transform \mathcal{D} of $Z + C'$ is simple normal crossing, *i.e.* we're planning on running the same argument as before but with \mathcal{S}_n instead of S , and C' instead of B . The singularities remain log-canonical so this time for ∂ the part of \mathcal{D} supported in $\rho^{-1}(C')$, with F again denoting the rest, there is a divisor E contracted by ρ such that,

$$(K_{\mathcal{X}} + \partial) = \rho^*(K_{\mathcal{S}_n} + C') + E$$

By 1.2.6.(c), we again have a closed current T_n on \mathcal{S}_n arising from big discs, which we may suppose to have origins bounded away from $Z + C'$, and whence,

$V \cdot T_n \geq 0$ for every divisor V supported in $Z + C'$. Next we take a logarithmic derivative, T'_n on $\pi : \mathbb{P}(\Omega_{\mathcal{X}}^1(\log \mathcal{D})) \rightarrow \mathcal{X}$ with L the tautological bundle, and, although the origins of the discs can be close to C , we still have the bound $L \cdot T'_n \leq F \cdot T_n$ because close to C 1.19 applies rather than 1.23, so, if we write,

$$T'_n = \mathbb{I}_{\pi^{-1}(\mathcal{D})} T'_n + R_n$$

then $L \cdot \mathbb{I}_{\pi^{-1}(\mathcal{D})} T'_n \geq (K_{\mathcal{X}} + \mathcal{D}) \cdot \mathbb{I}_{\mathcal{D}} T_n$ for the same reason as 2.2.1, while by 2.1.4, (the possibly nil) current R_n is supported in at most fibres of π . As such,

$$0 \geq \left(K_S + \left(1 - \frac{1}{n}\right) C + C' \right) \cdot T$$

and the images of the T_n in the homology of S are constant, so taking $n \rightarrow \infty$ we get exactly the same contradiction as before. \square

2.3 Kobayashi metric

Since we'll have a certain need for a good definition of tangent bundle, let's start with $(\mathcal{X}, \mathcal{D})$ a smooth champ de Deligne-Mumford with simple normal crossing boundary, and recall,

Definition 2.3.1. *Let $(\mathcal{X}, \mathcal{D})$ be as above, then for $x \in \mathcal{X} \setminus \mathcal{D}$, $t \in T_{\mathcal{X}}(-\log \mathcal{D}) \otimes \mathbb{C}(x)$,*

$$\text{Kob}(t)_{\mathcal{X} \setminus \mathcal{D}, x} = \inf \left\{ \frac{1}{|R|} \mid f : (\Delta, 0) \rightarrow (\mathcal{X} \setminus \mathcal{D}, x) : f_* \left(\frac{\partial}{\partial z} \right) = Rt \right\}.$$

A useful tool for the calculation of which is,

Fact 2.3.2. Let $x \in \mathcal{X} \setminus \mathcal{D}$ be as above, and suppose that every sequence of pointed discs in $\mathcal{X} \setminus \mathcal{D}$ with origin x admits a subsequence converging in \mathcal{X} uniformly on compact sets of Δ , then for any (étale) neighbourhood $V \rightarrow \mathcal{X} \setminus \mathcal{D}$ of x , there is a constant $c(x, V) > 0$ such that,

$$c(x, V) \text{Kob}_{V, x} \leq \text{Kob}_{\mathcal{X} \setminus \mathcal{D}} \leq \text{Kob}_{V, x}$$

Proof. The inequality on the right is automatic, *i.e.* by definition the Kobayashi metric increases under any mapping to $\mathcal{X} \setminus \mathcal{D}$. As to the left, if it fails, we may suppose that there are a sequences of discs, $f_n : (\Delta, 0) \rightarrow (\mathcal{X} \setminus \mathcal{D}, x)$, $g_n : (\Delta, 0) \rightarrow V$, with:

$$(f_n)_* \left(\frac{\partial}{\partial z} \right) = R_n (g_n)_* \left(\frac{\partial}{\partial z} \right), \quad R_n \rightarrow \infty$$

with the g_n 's as close to an extremal disc in the given direction as we please. By hypothesis, however, after subsequencing, f_n converge to a disc. Whence there is some $0 < r < 1$, such that for all sufficiently large n , $f_n(\Delta(r)) \subset V$, so R_n is at most $(1 + \varepsilon)/r$, for $\varepsilon > 0$ only depending on how far the g_n are from being extremal. \square

Let us also observe that this continues to work for $x \in \mathcal{D}$, to wit:

Fact 2.3.3. Everything as in 2.3.2, except we suppose the aforesaid convergence for pointed discs in $\mathcal{X} \setminus \mathcal{D}$ with origin in some relative compact $U \subset \mathcal{X} \setminus \mathcal{D}$ to a disc in \mathcal{X} , then for any (étale) neighbourhood $V \rightarrow \mathcal{X}$ of the closure of U , there is a constant $c(U, V) > 0$ such that for all $u \in U$,

$$c(U, V) \text{Kob}_{V \setminus \mathcal{D}, u} \leq \text{Kob}_{\mathcal{X} \setminus \mathcal{D}, u} \leq \text{Kob}_{V \setminus \mathcal{D}, u}$$

Proof. As above, but take the origins in U . □

and, in fact, this really has nothing to do with \mathcal{D} being a divisor, *i.e.*

Fact 2.3.4. The only place in the above where we're using that \mathcal{D} is a (simple normal crossing) divisor or \mathcal{X} is smooth is in order for $T_{\mathcal{X}}(-\log \mathcal{D})$ to be a bundle, and whence the infinitesimal form, 2.3.1, of the Kobayashi semi-distance is well defined. The integrated form of the distance is, however, always well defined, so, the above makes perfect sense even if \mathcal{X} is singular, or \mathcal{D} is any sub champ, in the integrated form, or, even infinitesimal form if the only condition is $\mathcal{X} \setminus \mathcal{D}$ smooth and irrespective of the co-dimension of \mathcal{D} .

There are, however, differences in the behaviour of the Kobayashi metric according to the co-dimension of the boundary, but in the presence of completeness such as 1.3.9.(c), or 2.1.2.(c), what the difference may be is by 2.3.2-2.3.4 a wholly local problem. As such let us discuss some such problems by way of,

Remark 2.3.5. Suppose the boundary were indeed a simple normal crossing divisor, then, by the uniformisation theorem, for V as above sufficiently small $\text{Kob}_{V \setminus \mathcal{D}}$ is complete and comparable above and below to 1.21. Thus,

(a) Hypothesis as in 2.3.3 and say $O \subset U$ open, $\text{Kob}_{\mathcal{X} \setminus \mathcal{D}}$ is complete in O . At the other extreme, if \mathcal{D} were a smooth point, p , say, to avoid notational confusion, of \mathcal{X} , then completeness is impossible since no matter how small V , $\text{Kob}_{V \setminus p} = \text{Kob}_V$, and we have a further confirmation of the need to fill the boundary in 2.1.2. On the other hand if it were an elliptic Gorenstein singularity, then for V small, $\text{Kob}_{V \setminus p}$ is complete. Indeed, up to the action of a finite group, the cases are two: the minimal smooth resolution of the singularity has an exceptional divisor, E , an elliptic curve, or a cycle of rational curves, which arise as cusps on quotients of the ball, respectively the bi-disc, so:

(b) Under the hypothesis of 2.3.4 and say \mathcal{D} a point p , then if \mathcal{X} is smooth at p , $\text{Kob}_{\mathcal{X} \setminus p}$ is never complete in the complement of any open by p . If, however, p is elliptic Gorenstein, then $\text{Kob}_{\mathcal{X} \setminus p}$ would be complete in a neighbourhood of p , and for $\rho : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ a minimal resolution of the singularity, $\rho^* \text{Kob}_{\mathcal{X} \setminus p}$ is compatible above and below with 1.21.

Finally, consider a case where there is a difference between 1.3.9, and 2.1.2, *e.g.* X of the former is a surface with a Duval singularity, p , obtained by contracting some divisor F on the minimal smooth resolution, Y , and X is the minimal model of $Y \setminus F$. As such, in 2.1.2, we should look to the Vistoli cover, 2.1.1, $\mathcal{X} \rightarrow X$, around which p is smooth, and everything is as in (b) above. In 1.3.9,

however, we're really working with the Kobayashi distance in X , which may not have an infinitesimal form, and is locally, whence also globally under hypothesis such as 2.3.4, different from the Kobayashi distance in \mathcal{X} .

The upshot of which is that the definitions 1.3.9, and 2.1.2 are best possible, the mechanism employed in, for example, [G], to create “counterexamples” to the Bloch principle is clear, while 2.1.2 implies that Kobayashi's intrinsic metric on the canonical model (S, B) of a quasi-projective surface with elliptic Gorenstein locus P is complete on $\mathcal{S} \setminus \{B \cup P\}$, and compatible above and below to the Kähler-Einstein metric around $B \cup P$ provided that GG holds, and we're bounded away from the exceptional set Z . Whence, it only remains to investigate the degeneration around Z . To this end, modulo eschewing any manifestly unnecessary blow ups off Z , let $\rho : X \rightarrow S$ be as in the proof of 2.2.4, with, $(X, D) \xrightarrow{p} (S_0, B_0) \xrightarrow{q} (S, B)$ the factorisation of ρ through a minimal modification of any elliptic Gorenstein singularities, *i.e.* modulo change of notation, (\tilde{S}, \tilde{B}) of 2.1.4. Further let, D, ∂, ζ be the total transforms of $B_0 + q^*Z, B_0, Z$, respectively, in X , and, again in X , \tilde{B} , the proper transform of B_0 , with \tilde{Z} defined by $D = \partial + \tilde{Z}$. Unravelling all of 1.4.1-1.4.3, according to a fashion suggested by 2.2.1 amounts for $f : \Delta \rightarrow X \setminus \partial$ and $f' : \Delta \rightarrow \mathbb{P}(\Omega_X^1(\log D))$ to the inequality,

$$\log \left[\frac{\|f_* (\frac{\partial}{\partial z})\| \mathbb{I}_{\tilde{Z}}}{|\log \mathbb{I}_{\zeta}|} \right](0) + \int_{\Delta(r)} (f')^* c_1(\bar{L}) - f^* c_1(\tilde{Z}) \leq \log \frac{l^c(r)}{r} \cdot \int_{\Delta(r)} f^* \omega \quad (2.3)$$

where the implied metricisation, $\| \cdot \|$ of $\Omega_X^1(\log D)$ is smooth around components of ζ , complete around components in \tilde{B} , the distance functions to divisors are appropriately bounded above as post 1.22, and l^c is the length of the boundary of $\Delta(r)$ in a complete metric for $X \setminus D$. By 2.3.2-2.3.4, and 1.16 we know exactly what the Kobayashi metric looks like unless there are maps f_n with unbounded Nevanlinna area for any r , and origins $f_n(0)$ close to ζ . In this scenario an easy case to distinguish- since it gives a better bound almost trivially- is when presented with such a sequence of discs f_n with large derivatives at the origin, for some r outside a set of finite hyperbolic measure, one finds the condition,

$$\limsup_n \left(\int_{\Delta(r)} f_n^* \omega_{X \setminus \tilde{B}} \right)^{-1} \int_{\Delta(r)} f_n^* c_1(\bar{L}) = \infty \quad (2.4)$$

where the metric $\omega_{X \setminus \tilde{B}}$ is complete on $X \setminus \tilde{B}$. Should this occur, then, apart from the term at the origin, the tautological degree dominates everything in 2.3 except possibly some, $(1 + \delta) \log \log \log \mathbb{I}_{\zeta}$ at the origin, for any $\delta > 0$, arising from the error in 1.22, and 1.2.5, so that in this case, there must be a constant $C(\delta) > 0$, such that for n sufficiently large, we have the bound:

$$\left[\frac{\|f_{n*} (\frac{\partial}{\partial z})\| \mathbb{I}_{\tilde{Z}}}{|\log \mathbb{I}_{\zeta}| (\log |\log \mathbb{I}_{\zeta}|)^{1+\delta}} \right] (0) \leq C(\delta) \quad (2.5)$$

As such we may otherwise suppose for any sequence of discs that we have to deal with (normalised not quite as in 1.5, but rather on dividing by the Nevanlinna

area of $\omega_{X \setminus \tilde{B}}$) that as per 1.2.6, we'll be able to lift any limiting current T of discs to a closed positive derivative T' , or more precisely, since T must be supported on ζ , a current away from $\zeta \cap \tilde{B}$, while at such singularities the background metric on fibres of the projective tangent space $P := \mathbb{P}(\Omega_X^1(\log D))$ may be degenerate, *i.e.* $c_1(\overline{L})$ in the above metricisation, so around such points we only suppose that T is finite valued on forms which can be bounded above and below by a linear combination of $c_1(\overline{L})$, and $\omega_{X \setminus \tilde{B}}$. Thus, we're varying the proof of 2.2.1 in order to get a bound on the degeneracy of the Kobayashi metric which is very close to 2.5. To this end for $C \subset \zeta \subset D$ as towards the end of the proof of 2.2.1, and $t \geq 0$ the metricisation of the residual section $S_C \subset P_C$ defined by the choice of norm on $\Omega_X^1(\log D)$ consider the function,

$$\phi_C := \max\{-\log |\log \mathbb{I}_C|, \log t\} \quad (2.6)$$

where in an abuse of notation we identify t with a lifting to P , since all liftings result in the same behaviour close to S_C , *i.e.* comparable psh. functions achieving the value $-\infty$ only on S_C , and, supposing suitable normalisation be it of the supremum whether of \mathbb{I}_C or t an additional lower bound,

$$dd^c \phi_C + N \omega_P \geq 0 \quad (2.7)$$

on the whole of P for some $N \in \mathbb{N}$, where, $\omega_P = c_1(\overline{L}) + N \omega_{X \setminus \tilde{B}} \geq 0$, for the same appropriately large N . Now, for much the same reasoning as that which leads from 2.4 to 2.5, we may without loss of generality suppose that for r outside a set of finite hyperbolic measure,

$$\liminf_n \left(\int_{\Delta(r)} f_n^* \omega_{X \setminus \tilde{B}} \right)^{-1} f_n^* \phi_C(0) > -\infty \quad (2.8)$$

since otherwise, we can subsequence to obtain for any $\varepsilon > 0$ and n sufficiently large, the lower bound,

$$\int_{\Delta(r)} (f'_n)^* c_1(\overline{L}) \geq \int_{\Delta(r)} (f_n)^* \omega_{X \setminus \tilde{B}} + \varepsilon (f_n^* \phi_C)(0) \quad (2.9)$$

which, up to an adjustment for $\exp(\varepsilon \phi)$ in the origin, leads to a similar bound to 2.5 for the derivative at the origin, the exact form of which we eschew, since, like 2.5, it's better than our final bound, 2.3.6, which will correspond to the case $\varepsilon = 1$. As such, supposing 2.8, the non-negative absolutely continuous 1-form defined by the left hand side of 2.7 has, for every n , a Nevanlinna integral of the same order as that of $\omega_{X \setminus \tilde{B}}$, independently be it of n or r . In particular, even though there is no algebraic variety on which the $c_1(E_n)$ of 2.2.2 would be equal to $-dd^c \phi_C$ Lebesgue almost everywhere (equivalently we would like to replace $|x|^{1/n}$ in op. cit. by $|\log |x||^{-1}$) the proof of 2.2.3 remains valid provided that,
(a) The Nevanlinna area in the complete metric on $X \setminus D$ remains comparable to that of $\omega_{X \setminus \tilde{B}}$, whence guaranteeing analogous conditions to 2.2.3.

(b) We change appropriately the adjunction formula for the chern class of a residual section according to the above metricisation of L .

The first point we have already encountered how to proceed, *i.e.* should this fail then by 1.23 we could replace ϕ_C in 2.9 by $\log |\log \mathbb{I}_\zeta|$, and obtain a better estimate on the degeneration of the Kobayashi metric than that which will ultimately be proposed. As to the second item, the problem is only relevant at the finite set $\zeta \cap \tilde{B}$, at which we can model $\omega_{X \setminus \tilde{B}}$ on a complete Kähler-Einstein metric of curvature -1 , with a Kähler coordinate along the component of ζ , and from which a metric as prescribed on $\Omega_X^1(\log D)$, satisfying adjunction whereby, the metricisation of $\mathcal{O}(K_X + D)$ as a sub-bundle of $\Omega_X^1(\log D)|_C$ is, up to a bounded continuous function, the restriction of the determinant. Consequently, we may proceed as per 2.2.1 to achieve,

$$\int_{\Delta(r)} (f'_n)^* c_1(\overline{L}) - (f_n)^* c_1(\overline{K_X + D}) \geq (f_n^* \phi)(0) + \circ \left(\int_{\Delta(r)} (f_n)^* \omega_{X \setminus \tilde{B}} \right) \quad (2.10)$$

Writing $D = \partial + \tilde{Z}$, and taking \overline{E} as in 2.2, albeit smoothly metricised, we have for $(X, D) \xrightarrow{p} (S_0, B_0) \xrightarrow{q} (S, B)$, the above factorisation of ρ , and $K_{S_0} + B_0 = q^*(K_S + B)$ metricised either in the Kähler-Einstein metric or a model thereof,

$$c_1(\overline{K_X + \partial} - p^* \overline{K_{S_0} + B_0} - \overline{E}) = dd^c(\log \log^2 \mathbb{I}_{B_0} - \log \log^2 \mathbb{I}_{\tilde{B}} + \psi) \quad (2.11)$$

where ψ is bounded above and below, so, say, below by zero, whence:

$$\int_{\Delta(r)} (f_n)^* c_1(\overline{K_X + \partial}) \geq \int_{\Delta(r)} (f_n)^* c_1(\overline{K_S + B}) - \log f_n^* \left[\frac{|\log \mathbb{I}_{B_0}|}{|\mathbb{I}_E| \log \mathbb{I}_{\tilde{B}}|} \right] (0)$$

The intersections along \tilde{Z} in 2.10 and 2.3 cancel, we concede $\phi \geq -\log |\log \mathbb{I}_\zeta|$ to avoid a bound that might better that afforded by the supposition implying 2.10, use the ampleness of $K_S + B$ as in the proof of 2.2.4, and so as per 2.5,

Proposition 2.3.6. *Suppose the infinito property GG of 2.1.2.(d) holds for a quasi-projective algebraic surface, then on the canonical model (S, B) , of the necessarily general type pair, the Kobayashi-metric is bounded above by a constant times the Kähler-Einstein metric, and conversely, below, except around the (minimal) exceptional locus Z , where, notations as above, for every $\delta > 0$ there is a constant $c(\delta) > 0$ such that at worst:*

$$\rho^* \text{Kob}_{S \setminus \{B \cup P\}} \geq c(\delta) \left[\frac{\mathbb{I}_{\tilde{Z}} \mathbb{I}_E |\log \mathbb{I}_{\tilde{B}}|}{|\log \mathbb{I}_\zeta|^2 (\log |\log \mathbb{I}_\zeta|)^{1+\delta} |\log p^* \mathbb{I}_{B_0}|} \right] \tau$$

For τ a smooth metric on $\Omega_X(\log D)$ except around the components \tilde{B} of the proper transform of B_0 , *i.e.* of B and the elliptic Gorenstein singularities, P , where it is mildly degenerate being modelled on a complete metric, 1.21, around the said components.

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